Exponential Pareto Negative Binomial Distribution With Its Properties And Application

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Abstract : In this research, we consider certain results characterizing the generalization of Exponential Pareto and Negative Binomial Distribution through their distribution functions and asymptotic properties. The resulting Exponential Pareto Negative Binomial Distribution [EPNBD] was defined and some of its properties like moment generating function, survival rate function, hazard rate function and cumulative distribution function were investigated. The estimation of the model parameters was performed using maximum likelihood estimation method. The distribution was found to generalize some known distributions thereby providing a great flexibility in modeling symmetric, heavy tailed, skewed and bimodal distributions, the use of the new lifetime distribution was illustrated using failure time life data.

Keywords: Moment Generating Function, Survival Rate, Harzard rate, Exponential Pareto Negative Binomial Distribution, Maximum Likelihood Method.

1. Introduction

The Pareto distribution was originally proposed to model the unequal distribution of wealth since he observed the way that a larger portion of the wealth of any society is owned by a smaller percentage of the people. Ever since, it plays an important role in analysing a wide range of real-world situations, not only in the field of economics. Examples of approximately Pareto distributed phenomena may be found in sizes of sand particles and clusters of Bose-Einstein condensate close to absolute zero. There are several forms and extensions of the Pareto distribution in the literature. Pickands (1975) was the first to propose an extension of the Pareto distribution with the generalized Pareto (GP) distribution when analyzing the upper tail of a distribution function. Providing an empirical generalization of the aggregate market, the NBD describes the penetration and purchase incidence of a product category or a single brand and is underpinned by two assumptions about consumers' purchase incidence (Ehrenberg, 1959). Collectively, these two assumptions imply that each consumer has a steady long-run purchase probability or rate for a product category (e.g., a consumer buys ten times in a year), but there is no regular pattern in the purchase distribution over time (i.e., the ten purchases are randomly timed within the year). In recent years, various new classes of distributions have been introduced based on the Weibull distribution, which is one of the most widely used distributions for modelling lifetime data such as, the Weibull exponentiated (WE) distribution introduced by Mudholkar and Srivastava, the extended Weibull distribution proposed by Xie et al., (2002), the modified Weibull (MW) distribution introduced by Lai et al., (2003), the beta-Weibull (BW) distribution introduced by Famoye et al., (2005) the modified beta-Weibull (BW) distribution introduced by Silva et al., (2010) which generalizes the modified Weibull (MW), Weibull exponentiated (WE) and beta-Weibull (BW) distributions. Barreto-Souza et al., (2011) proposed the Weibull-geometric distribution obtained by compounding the Weibull and the geometric distributions. Oguntunde et al., (2015) presented a paper on a three parameters probability model, called Weibull-exponential distribution and was proposed using the Weibull Generalized family of distributions. A number of standard theoretical distributions have been found to be useful in the fields of insurance, engineering, medicine, economics and finance (among others).

2. Derivation and Development of Exponential Pareto- Negative Binomial Distribution

Kareema et al., (2013) gave the cdf $F(x) = \int_a^b f(x)dx$, $where\ b = \infty = \frac{1}{1-F^{\epsilon}(x)}$

$and\ \ F^{\epsilon}(x)\ \ is\ the\ cdf\ of\ any\ distribution\ to\ be\ compounded\ with\ parent\ distribution.$

In this case, our $F^{\epsilon}(x)$ is cdf of any parent distribution and f(x) is the pdf of exponential distribution from which the required Exponential-X Mixture distribution can be obtained.
Therefore,

$$F_{EPD}(x) = \int_0^{\frac{1}{1-F^{\epsilon}(x)}} f(x)dx,\ \ where\ f(x) = \lambda e^{-x\lambda}\ \ for\ \ x > 0 \quad (1)$$

$$F_{EPD}(x) = \int_0^{\left(\frac{\alpha}{x}\right)^{\theta}} \lambda e^{-x\lambda}\ dx = 1 - e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}} \quad (2)$$

The pdf of of exponential pareto can be obtained by

$$f_{EPD}(x) = \frac{d\left(1-e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)}{dx}$$

$$f_{EPD}(x) = \frac{\lambda\theta}{\alpha}\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}} \quad (3)$$

According to Cristiane et al., (2011), let $x_1, x_2 \dots x_z$ be a random sample from exponential pareto density function with scale parameter $\alpha, \lambda > 0$ shape parameter $\theta > 0$ and we assumed that the random variable z has a zero truncated negative binomial with $\beta \in (0,1) and\ pmf = \beta^z \binom{s+z-1}{z}[(1-\beta)^{-s} - 1]^{-1}$, the pdf of X can be obtained as:

$f_{EPD}(x) =$

$$\beta^z \binom{s+z-1}{z} [(1-\beta)^{-s}-1]^{-1} . \frac{\lambda\theta}{\alpha}\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}$$
(4)

Use the relation $(a+b)^{-r} = \sum_{k=0}^{\infty} \binom{-r}{k} a^{-r-k} b^k$ and $(1-s)^{-r} = \sum_{k=0}^{\infty} \binom{-r}{k} s^k = (-1)\sum_{k=0}^{\infty} \binom{-r}{k} s^k$ and it can be express as $(-1)^k \frac{(r+(k-1))(r+(k-2))\ldots(r+(k-k)))}{k!} = (-1)^k \binom{r+k-1}{k}$. (5)

By applying the relation, the $f_{EPD}(x)$ can be simplified to arrive at (6)

$$f_{EPNBD}(x;\tau) = \frac{\lambda\theta s\beta}{\alpha[(1-\beta)^{-s}-1]} \left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}} \left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-(s+1)} \quad (6)$$

By integrating (6) we arrive at the cumulative distribution of Exponential Pareto Negative Binomial Distribution $F_{EPNBD}(x;\tau)$ where $\tau = (s,\beta,\theta,\alpha,\lambda)$

$$F_{EPNBD}(x;\tau) = \frac{\lambda\theta s\beta}{\alpha[(1-\beta)^{-s}-1]} \int_0^x \left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}} \left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-(s+1)} dx \quad (7)$$

$$= \frac{s\beta}{[(1-\beta)^{-s}-1]} \int_0^x e^{-y}(1-\beta e^{-y})^{-(s+1)} dy = \frac{s\beta}{[(1-\beta)^{-s}-1]} \int_0^x \frac{e^{-y}}{(1-\beta e^{-y})^{-(s+1)}} dy \quad (8)$$

$$= \frac{1}{[(1-\beta)^{-s}-1]} \left[-\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-s} + (1-\beta)^{-s}\right]$$
(9)

$$F_{EPNBD}(x;\tau) = \frac{1}{[(1-\beta)^{-s}-1]} \left[(1-\beta)^{-s} - \left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-s}\right] \quad (10)$$

$$F_{EPNBD}(x;\tau) = \frac{\left[(1-\beta)^{-s} - \left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-s}\right]}{[(1-\beta)^{-s}-1]}$$
(11)

By following the method of expansion of the pdf, the cdf can be expresses as:

$F_{EPNBD}(x;\tau) = \sum_0^{\infty} w_k G_{\lambda(k+1)\theta}(x)$, where

$$G_{\lambda(k+1)\theta}(x) = 1 - e^{-\lambda(k+1)\left(\frac{x}{\alpha}\right)^{\theta}} \quad (12)$$

Moment

By following the expansion of the pdf of EPNBD

$$= \sum_{k=0}^{\infty} \frac{(s+1)}{[(s+1)+k]} \binom{(s+1)+k}{k} \left\{\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right\} \left\{1 - \left(\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{1-1}\right\}^k$$

$$= \sum_{k=0}^{\infty} \binom{s+k}{k} \left(\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^k$$
(13)

by applying equation (13) in equation(6),we have

$$f_{EPNBD}(x;\tau) = \frac{\lambda\theta s\beta}{\alpha[(1-\beta)^{-s}-1]} \sum_{k=0}^{\infty} \binom{s+k}{k} \beta^k e^{-\lambda(k+1)\left(\frac{x}{\alpha}\right)^{\theta}}$$
(14)

$$= \frac{\lambda\theta s}{\alpha[(1-\beta)^{-s}-1]} \sum_{k=0}^{\infty} \binom{s+k}{k} \left(\frac{x}{\alpha}\right)^{\theta-1} \beta^{k+1} e^{-\lambda(k+1)\left(\frac{x}{\alpha}\right)^{\theta}}$$
(15)

$$f_{EPNBD}(x;\tau) = \frac{\lambda\theta s}{\alpha[(1-\beta)^{-s}-1]} \sum_{k=0}^{\infty} \binom{s+k}{k} \left(\frac{x}{\alpha}\right)^{\theta-1} \beta^{k+1} e^{-\lambda(k+1)\left(\frac{x}{\alpha}\right)^{\theta}} \quad (16)$$

$$= \sum_{k=0}^{\infty} \frac{s\beta^{k+1}\binom{s+k}{k}}{(k+1)[(1-\beta)^{-s}-1]} g_{\alpha(k+1),\theta}(x)$$
(17)

$$E(x^r) = \int_0^{\infty} f_{(EPD)}(x;\beta,\lambda,\theta,\alpha)\, dx = \int_0^{\infty} x^r \frac{\lambda\theta}{\alpha}\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}} dx$$
(18)

$$E(x^r) = \sum_{k=0}^{\infty} w_k \left(\frac{\alpha}{\sqrt[\theta]{\lambda(k+1)}}\right)^r \Gamma\left(\frac{r}{\theta}+1\right)$$
(19)

The factorial moment of EPNBD is given by equation (19), foe r = 1,2,3,4

$$E(x) = \sum_{k=0}^{\infty} w_k \left(\frac{\alpha}{\sqrt[\theta]{\lambda(k+1)}}\right) \Gamma\left(\frac{1}{\theta}+1\right) \quad r = 1$$
(20)

$$E(x^2) = \sum_{k=0}^{\infty} w_k \left(\frac{\alpha}{\sqrt[\theta]{\lambda(k+1)}}\right)^2 \Gamma\left(\frac{2}{\theta}+1\right)$$
(21)

$$E(x^3) = \sum_{k=0}^{\infty} w_k \left(\frac{\alpha}{\sqrt[\theta]{\lambda(k+1)}}\right)^3 \Gamma\left(\frac{3}{\theta}+1\right)$$
(22)

$$E(x^4) = \sum_{k=0}^{\infty} w_k \left(\frac{\alpha}{\sqrt[\theta]{\lambda(k+1)}}\right)^4 \Gamma\left(\frac{4}{\theta}+1\right)$$
(23)

Variance = $E(x^2) - (E(x))^2$

$$= \sum_{k=0}^{\infty} w_k \left(\frac{\alpha}{\sqrt[\theta]{\lambda(k+1)}}\right)^2 \Gamma\left(\frac{2}{\theta}+1\right) - \sum_{k=0}^{\infty} w_k \left(\frac{\alpha}{\sqrt[\theta]{\lambda(k+1)}}\right) \Gamma\left(\frac{1}{\theta}+1\right) \quad (24)$$

The central moments μ_s and the cumulants k_s of x can be obtained directly from the ordinary moment by:

$$\mu_p = \sum_{r=0}^{\infty}(-1)^r \binom{p}{r} (\mu')^p \mu_{p-r}' \text{ and } k_p = \mu_p' - \sum_{r=1}^{p-1} k_p\, \mu_{p-r}' \quad (25)$$

$$k_1 = \mu'_1,\ k_2 = \mu'_2 - (\mu'_1)^2,\ k_3 = \mu'_3 - 3\mu_2'\mu_1' + 2(\mu'_1)^3 \quad (26)$$

$$k_4 = \mu'_2 - 4\mu_3'\mu_1' - 3(\mu'_2)^2 + 12\mu'_2(\mu'_1)^2 - 6\mu'^4_1$$
(27)

Skewness = $\frac{k_3}{(k_2)^{\frac{3}{2}}}$; kurtosis = $\frac{k_4}{k_2{}^2}$

$$\text{Skewness} = \frac{Ex^3 - 3ExEx^2 + 3(Ex)^2Ex - (Ex)^3}{(Ex^2-(Ex)^2)^{\frac{3}{2}}} = \frac{Ex^3 - 3ExEx^2 + 2(Ex)^3}{(Ex^2-(Ex)^2)^{\frac{3}{2}}}$$
(28)

$$\text{Kurtosis} = \frac{Ex^4 - 4ExEx^3 + 6\mu^2Ex^2 - 3(Ex)^4}{(Ex^2-(Ex)^2)^2}$$
(29)

Reliability

The reliability function R(x), which is the probability of an item not failing prior to time t, is defined by R(t) = 1 − F(t).

$$R(t) = 1 - \frac{\left[(1-\beta)^{-s} - \left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-s}\right]}{[(1-\beta)^{-s}-1]}$$
(30)

Hazard Function

Hazard function is represented in mathematical form as

$$H(x) = \frac{f_{EPNBD}(x;\tau)}{R(t)} = \frac{\frac{\lambda\theta s\beta}{\alpha[(1-\beta)^{-s}-1]}\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-(s+1)}}{1-\frac{\left[(1-\beta)^{-s}-\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-s}\right]}{[(1-\beta)^{-s}-1]}} \quad (31)$$

$$H(x) = \frac{\frac{\lambda\theta s\beta}{\alpha[(1-\beta)^{-s}-1]}\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-(s+1)}}{\frac{[(1-\beta)^{-s}-1]-\left[(1-\beta)^{-s}-\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-s}\right]}{[(1-\beta)^{-s}-1]}} \quad (32)$$

$$= \frac{\lambda\theta s\beta\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-(s+1)}}{\alpha[(1-\beta)^{-s}-1]} \; x \; \frac{[(1-\beta)^{-s}-1}{(1-\beta)^{-s}-(1-\beta)^{-s}-1+\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)} \quad (33)$$

$$= \frac{\lambda\theta s\beta\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-(s+1)}}{\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-s}-1} \quad (34)$$

3. Method of Maximum likelihood

This method is used to estimate the parameters of the newly proposed model, this was done as follows

$$\mathrm{L}[f_{EPNBD}(x;\tau)] = \prod_{i=1}^{n}\frac{\lambda\theta s\beta}{\alpha[(1-\beta)^{-s}-1]}\left(\frac{x}{\alpha}\right)^{\theta-1} e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\left(1-\beta e^{-\lambda\left(\frac{x}{\alpha}\right)^{\theta}}\right)^{-(s+1)}$$

$$\log \mathbf{L}[f_{EPNBD}(x;s,\beta,\theta,\alpha,\lambda)] = n\log(\lambda) + n\log(\beta) + n\log(\theta) + n\log(s) - n\log(\alpha) + (\theta-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\alpha}\right) - \lambda\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\theta} - (s+1)\sum_{i=1}^{n}\log\left(1-\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}\right) - \log[(1-\beta)^{-s}-1]$$

$$\frac{\partial \log \mathrm{L}f_{EPNBD}(x;s,\beta,\theta,\alpha,\lambda)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\theta} + (s+1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\alpha}\right)^{\theta}\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}}{1-\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}} \quad (35)$$

$$\frac{\partial \log \mathrm{L}f_{EPNBD}(x;s,\beta,\theta,\alpha,\lambda)}{\partial s} = \frac{n}{s} - \sum_{i=1}^{n}\log\left(1-\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}\right) + \frac{(1-\beta)^{-s}\log(1-\beta)}{[(1-\beta)^{-s}-1]} \quad (36)$$

$$\frac{\partial \log \mathrm{L}f_{EPNBD}(x;s,\beta,\theta,\alpha,\lambda)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\log\left(\frac{x_i}{\alpha}\right)\lambda\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\theta}\log\left(\frac{x_i}{\alpha}\right) + (s+1)\sum_{i=1}^{n}\frac{\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}\log\left(\frac{x_i}{\alpha}\right)\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}}{1-\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}} \quad (37)$$

$$\frac{\partial \log \mathrm{L}f_{EPNBD}(x;s,\beta,\theta,\alpha,\lambda)}{\partial\beta} = \frac{n}{\beta} + (s+1)\sum_{i=0}^{n}\log\left(1-\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}\right) - \frac{s(1-\beta)^{-(s+1)}}{[(1-\beta)^{-s}-1} \quad (38)$$

$$=\frac{\partial \log \mathrm{L}f_{EPNBD}(x;s,\beta,\theta,\alpha,\lambda)}{\partial\alpha} = \frac{n}{\alpha} + n(\theta-1)\log(1/\alpha) + \sum_{i=1}^{n}\lambda\frac{\theta x_i}{\alpha^2}\left(\frac{x_i}{\alpha}\right)^{\theta-1} - (s+1)\sum_{i=1}^{n}\frac{\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}\lambda\frac{\theta x_i}{\alpha^2}\left(\frac{x_i}{\alpha}\right)^{\theta-1}}{\left(1-\beta e^{-\lambda\left(\frac{x_i}{\alpha}\right)^{\theta}}\right)} \quad (39)$$

4. Data Analysis and Application of EPNBD to failure time data

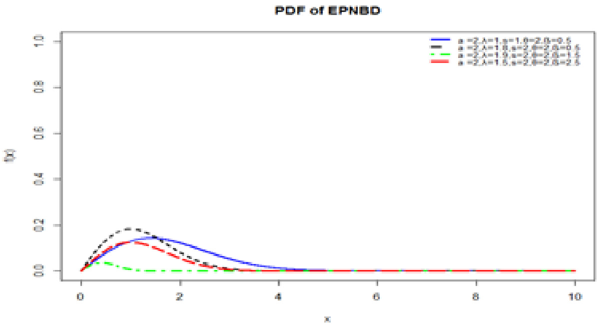

***Fig 1**: Probability Density of Exponential Pareto Negative Binomial (EPNBD)*

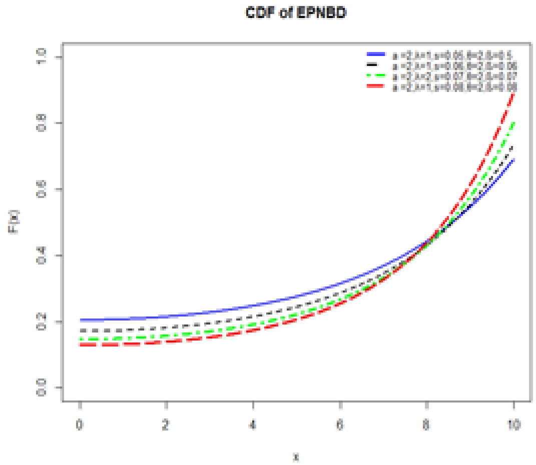

***Fig 2**: CDF of Exponential Pareto Negative Binomial Distribution*

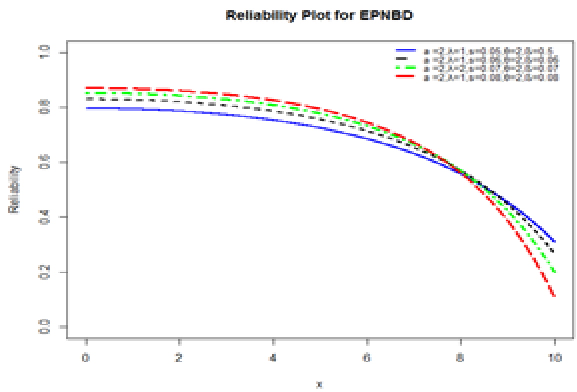

***Fig 3**: The Reliability Plot for Exponential Pareto Negative Binomial*

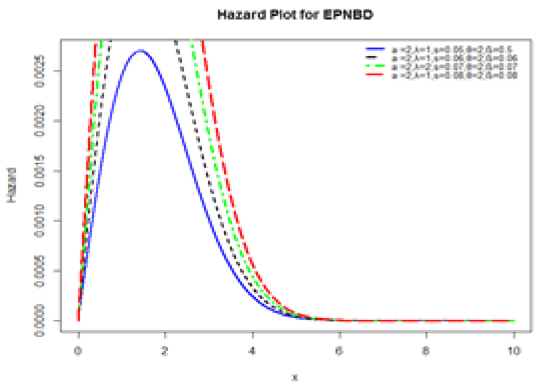

Fig 4: Hazard Plot for Exponential Pareto Negative Binomial Distribution

From the Fig 1, we can say that the Probability Density for the Exponential Pareto Negative Binomial (EPNBD) is heavily tailed, highly skewed and has a mode. For this reasons, we can deduce that it can be used to model the data with heavily tailed and highly skewed distribution, thus making it to model dynamical system whose outcome does not follow normal distribution.

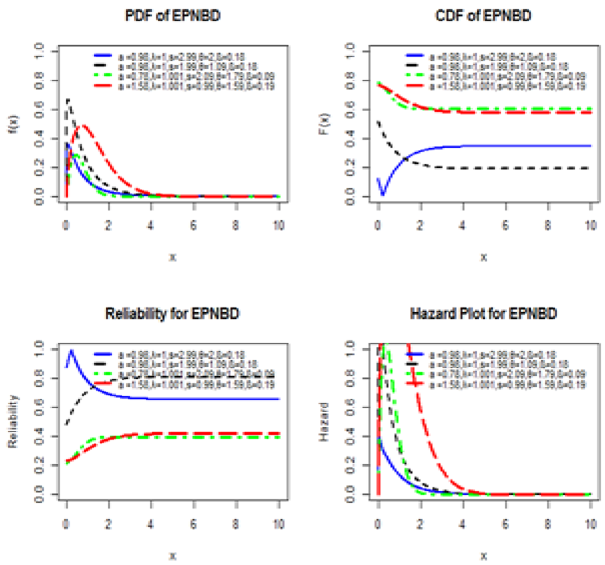

***Fig 5**: The Plots for Exponential Pareto Negative Binomial*

From the Fig 5 above, we can say that the Probability Density for the Exponential Pareto Negative Binomial (EPNBD) is heavily tailed, highly skewed and has a mode.

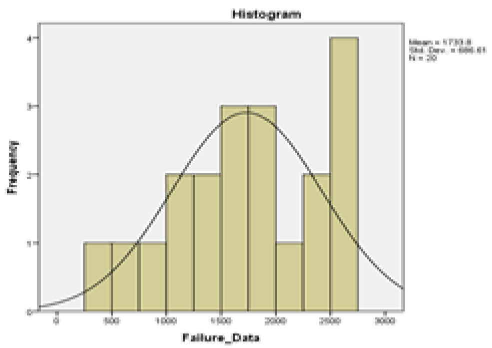

***Fig 6**: Investigating the Distribution of Data on Life test (Failure data)*

From the fig 6, we can also say that the data is not normal, heavily tailed, highly skewed, thus it follows Exponential Pareto Negative Binomial Distribution

Comparative Criteria

***Table 1**: Shows comparative criteria for Exponential Pareto Negative Binomial Distribution [EPNBD] and Weibull Negative Binomial (WNBD)*

	Exponential Pareto Negative Binomial Distribution (EPNBD)		Weibull Negative Binomial (WNBD)	
Parameters	Estimate	Standard Error	Estimate	Standard Error
λ	9.597e-07	3.997e-04		
β	1.645e+00	2.010e-03	1.291e+00	7.829e-04
θ	1.114e+03	1.673e-07	1.292e-03	3.081e-02
S	4.880e+00	2.010e-03	2.795e+00	5.531e-01
α	1.825e+04	1.199e-03	1.950e+01	2.618e-01
	AIC =-2074203		AIC = 774.623	

Examining the table above, it was discovered that the newly developed model [EPNBD] perform better than the WNBD in terms of the value obtained after subjecting our data on cooling time of a failure system to the two models. Since the AIC of EPNBD perform better than WNBD, then we say that EPNBD is much more consistence than WNBD. Expected time to system failure E(t) =5058.1938 hrs ,this means the average time that the cooling system is expected to operate before failure is 2537.61hrs.

5. Conclusion

In research, mixing distribution has played a vital role in tracking pattern of failure time of cooling system. It is on this note that we apply our developed Exponentiated Pareto Negative Binomial Distribution (EPNBD) to track the failure time of a cooling system in order to determine its performance vis-a-viz efficiency, consistency, stability and flexibility. A detailed study on the mathematical properties of the new distribution is presented. The research established that there are ways in which we can model non normal data effectively without concealing the richness of the information. By compounding two or more probability distributions, we get the corresponding hybrid distribution with increased number of parameters which is believed to give the newly compounded distribution more flexibility, consistency, stability, sufficiency uniqueness and wider application compare to the existing Weibull Negative Binomial Distribution. It is on this note that the significance of this distribution can be fetched in modeling countless of random phenomena, epidemiological and economic studies.

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