Some Characterizations of Lie And Jordan Ideals of Gamma Rings With Involution

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Abstract: In this article, we have developed some characterizations of Lie and Jordan ideals of simple and semi-prime gamma rings by means of involution mapping. Several properties of arbitrary gamma rings related to this work are also established here.

Keywords: Centre of a gamma ring, Involution gamma ring, Lie and Jordan ideals, Semi-prime gamma rings.

1. Introduction

Nobusawa (1964) first introduced the concepts of a Γ-ring in 1964 and his definition was generalized by Barnes (1966). R. Awtar (1976) discussed on Jordan structure in semi-prime ring and obtained some interesting results by setting Jordan ideals of semi-prime rings. P.M. Cohn (1973) found some fruitful solutions of all symmetric elements of prime rings with involution which are nilpotent or regular. Ibraheem and Majeed (2019) introduced the definition of lie ideal on inverse semi-ring and found out several interesting results using lie structure. Lie and Jordan structures in classical ring theories was studied by I. N. Herstein (1969) and some significant results of Lie and Jordan Ideals of Simple Rings with involution was established by him. C. Lanski (1972) worked on rings with involution and solved some basic problems of these rings and its Lie and Jordan ideals. S. Montgomery (1974) also discussed on rings with involution and included some remarkable results of these rings and its ideals. Lie and Jordan structures in simple Gamma Rings were studied by Paul and Sabur (2010) and some structural solutions of these ideals in simple Gamma Rings were shown by them. Paul and Sabur (2012, 2013) also worked on Lie and Jordan structures in Simple Gamma Rings with involution and they developed some structural properties of Gamma Rings and its Lie and Jordan Ideals. In the classical ring theories Lie and Jordan ideals of rings with involution was studied by I. N. Herstein (1976). In this study, we have generalized the results of I. N. Herstein (1976) into Lie and Jordan ideals of Gamma Rings with involution. The results which are found in Paul and Sabur (2012, 2013) are not appeared in this article.

2. Preliminaries

2-torsion free Gamma Ring 2.1. M is called a 2-torsion free Γ-ring if $2x = 0$ implies $x = 0$ for each $x \in$ M.

Prime Gamma Ring 2.2. A Γ-ring M is a prime Γ-ring if whenever $a\Gamma M\Gamma b = 0$, a, b ∈ M then either $a = 0$ or $b = 0$.

Notation 2.3. If A is a subset of a Γ-ring M then $\bar{A}$ will denote the sub-Γ-ring of M generated by A.

All other preliminaries are in Paul and Sabur [10,12, 13].

3. Some characterizations of Lie and Jordan Ideals of Gamma Rings with Involution.

Definition 3.1. An element x of a Γ-ring M with involution I are called respectively symmetric and skew if $I(x) = x$ and $I(x) = -x$. .We shall consistently use the notation $S = \{\boldsymbol{x} \in M | I(x) = x\}$ for the set of symmetric elements of M and $K = \{x \in M | I(x) = -x\}$ for the set of skew elements of M. In addition, we shall often be concerned with the set of traces T in M defined by $T = \{x + I(x) | x \in M\}$ and set of skew traces $K_0 = \{x - I(x) | x \in M\}$ of M .Now it is clear that $T \subset S$ and $K_0 \subset K$.

Definition 3.2. An additive subgroup A of a Γ-ring M is called a Jordan sub- Γ-ring of M if $a, b \in A$ implies that $(a, b)_\alpha = a\alpha b + b\alpha a \in A,\ \alpha \in \Gamma$. If A is a Jordan sub- Γ-ring of M and if $B \subset A$ is an additive subgroup such that $a \in A,\ b \in B$ implies $a\alpha b + b\alpha a \in B,\ \alpha \in \Gamma$ then B is called a Jordan ideal of A. We can write this as: $B \subset A$ is a Jordan ideal of A if $(B, A)_\Gamma \subset A$. An additive subgroup L of M is called a Lie sub-Γ-ring of M if whenever $a, b \in L$ and $\alpha \in \Gamma$ then $[a, b]_\alpha = a\alpha b - b\alpha a \in L$. When L is a Lie sub-Γ-ring of M, and $U \subset L$ is an additive subgroup such that $a\alpha u - u\alpha a \in U$ for any $a \in L,\ u \in U$ and $\alpha \in \Gamma$, then we call U is a Lie ideal of L.

If for subsets X, Y of M we denote by $[X, Y]_\Gamma$, the additive subgroup generated by all $x\alpha y - y\alpha x$ where $x \in X,\ y \in Y$ and $\alpha \in \Gamma$, then we could restate that U is a Lie ideal of L by merely noting that this is equivalent to saying that $[U, L]_\Gamma \subset L$.

Definition 3.3. M is called a semi-prime Γ-ring if it has no non-zero nilpotent ideals. Semi-primeness, like primeness can be characterized in terms of elements of the Γ-ring. This characterization run as follows: M is semi-prime iff $a\Gamma M\Gamma a = 0$ with $a \in M$, forces $a = 0$.

Lemma 3.4. Suppose that M is any Γ-ring and $P \neq 0$ is a nil right ideal of M. If every $x \in P$ satisfies $(x\alpha)^n x = 0$ for fixed integer n, then M cannot be semi-prime.

Theorem 3.5. Suppose that M is a semi-prime, 2-torsion free Γ-ring and let $U \neq 0$ is a Jordan ideal of M. Then any non-zero ideal of M is contained in U.

The result is appeared in ([10], Theorem 3.4).

Theorem 3.6. Suppose that M is a simple Γ-ring of characteristic not 2. Then M is simple as a Jordan Γ -ring.

The result is appeared in ([10], Corollary 3.5).

Theorem 3.7. Let M be any Γ-ring and A is a sub-Γ-ring and also a Lie ideal of M. Then A contains the ideal $M\Gamma[A,A]_\Gamma \Gamma M$ of M. In particular, if M is semi-prime and Z is a center of M, then:
(i) If A is not commutative, any non-zero ideal of M is contained in A;
(ii) if A is commutative, then if $a \in A$ and $\alpha \in \Gamma$ we must have $a\alpha a \in Z$;
(iii) if A is commutative and M is 2-torsion free, then $A \subset Z$.

This result leads to ([10], Theorem 3.15).

Theorem 3.8. Suppose that U be a Lie ideal of a simple Γ-ring M. Then either $U \subset Z$ or $U \supset [M,M]_\Gamma$, except if M is of characteristic 2 and 4-dimensional over Z.

The result is appeared in ([11], Theorem 3.19).

Theorem 3.9. Suppose that T is an additive subgroup of a simple Γ-ring M such that $[T,[M,M]_\Gamma]_\Gamma \subset T$. Then either $T \subset Z$ or $T \supset [M,M]_\Gamma$ except if M is of characterestic 2 and 4-dimensional over Z.

A special case of Theorem 3.8, but one of great interest, is its.

Corollary 3.10. Let M be a simple Γ-ring and let U be a Lie ideal of $[M,M]_\Gamma$. Then either $U \subset Z$ or $U = [M,M]_\Gamma$, except if M is of characterestic 2 and 4-dimensional over Z.

The result is appeared in ([11], Theorem 3.18).

Lemma 3.11. Suppose that M is a semi-prime Γ-ring and let $a \in M$ be such that $a\alpha(a\alpha x - x\alpha a) = 0$ for all $x \in M$ and $\alpha \in \Gamma$ then $a \in Z$.

Proof. If $x, m \in M$, then $a\alpha(a\alpha(x\alpha m - x\alpha m)\alpha a) = 0$.
However $a\alpha((x\alpha m) - (x\alpha m))\alpha a$
$= a\alpha(x\alpha m) - (x\alpha a)\alpha m + (x\alpha a)\alpha m - (x\alpha m)\alpha a$
$= (a\alpha x)\alpha m - (x\alpha a)\alpha m + x\alpha(a\alpha m) - x\alpha(m\alpha a)$
$= (a\alpha x - x\alpha a)\alpha m + x\alpha(a\alpha m - m\alpha a)$.
Then $a\alpha(a\alpha(x\alpha m) - (x\alpha m)\alpha a)$
$= a\alpha(a\alpha x)\alpha m - a\alpha(x\alpha a)\alpha m + a\alpha x\alpha(a\alpha m - m\alpha a)$.
So that
$0 = (a\alpha(a\alpha x) - a\alpha(x\alpha a))\alpha m + a\alpha x\alpha(a\alpha m - m\alpha a)$.
Therefore $0 = a\alpha x\alpha(a\alpha m - m\alpha a)$. Thus we get $a\alpha x\alpha(a\alpha m - m\alpha a) = 0$ for every $x, m \in M$, that is, $a\Gamma M\Gamma(a\alpha m - m\alpha a) = 0$. But this gives $(a\alpha m - m\alpha a)\Gamma M\Gamma(a\alpha m - m\alpha a) = 0$. Since M is semi-prime, we conclude that $a\alpha m - m\alpha a = 0$. Therefore $a\alpha m = m\alpha a$ for all $m \in M$. Hence $a \in Z$.

Lemma 3.12. Suppose that P is a right ideal of a semi-prime Γ-ring M. Then the centre of $, Z(P) \subset Z$.

Proof. If $a \in Z(P)$ and $x \in M$ then since $a\alpha x \in P$, $a\alpha(a\alpha x) = (a\alpha x)\alpha a$, that is, $a\alpha(a\alpha x - x\alpha a) = 0$.By Lemma 3.11, we conclude that $a \in Z$. Hence $Z(P) \subset Z$.

Lemma 3.13. Suppose that M is any Γ-ring with involution I and $\bar{S}$ be the sub- Γ-ring of M. Then $\bar{S}$ is a Lie ideal of M.

Proof. If $s \in S$, $m \in M$ and $\alpha \in \Gamma$ then $s\alpha m - m\alpha s = (s\alpha m + I(m)\alpha s) - (m + I(m))\alpha s$, so is in $\bar{S}$. Since commutation is a Γ-derivation of M, we immediately get that $[\bar{S}, M]_\Gamma \subset \bar{S}$. Hence $\bar{S}$ is a Lie ideal of M.

As is clear from the proof just given, $\bar{T}$ is also a Lie ideal of M.
We see some consequences of this general fact that $\bar{S}$ is always a Lie ideal of M.

Theorem 3.14. Suppose that M is a semi-prime Γ-ring with involution I. Then either $S \subset Z$ or any non-zero ideal of M is contained in $\bar{S}$.

Proof. Since $\bar{S}$ is both a sub- Γ-ring and a Lie ideal of M, if $\bar{S}$ is not commutative we have by Theorem 3.7 (i) that any non-zero ideal of M is contained in $\bar{S}$. So we may suppose that $\bar{S}$ is commutative, in which case $\bar{S}$ is merely S. Also from theorem 3.7(iii) $S \subset Z$ and in any event if $a \in S$ then $a\alpha a \in Z, \alpha \in \Gamma$.

Let $W = \{x \in M | 2x = 0\}$; W is an ideal of M, is of characteristic 2 and $I(W) \subset M$. Let $S_W = S \cap W$; if $a \in S_W$ and $x \in W$ then $a\alpha(x + I(x)) = (x + I(x))\alpha a$. Therefore
$a\alpha x + a\alpha I(x) = x\alpha a + I(x)\alpha a$
$\Rightarrow a\alpha x - x\alpha a = I(x)\alpha a - a\alpha I(x)$
$\Rightarrow a\alpha x - x\alpha a + 2x\alpha a = I(x)\alpha a - a\alpha I(x) + 2a\alpha I(x)$,
since $2x\alpha a = 0$, $2a\alpha I(x) = 0$
$\Rightarrow a\alpha x + x\alpha a = a\alpha I(x) + I(x)\alpha a$
$\Rightarrow a\alpha x + x\alpha a = I(a)\alpha I(x) + I(x)\alpha I(a)$, since $I(a) = a$.
Hence $a\alpha x + x\alpha a = I(a\alpha x + x\alpha a)$. That is, $a\alpha x + x\alpha a \in S_w$ and commutes with a. Now $a\alpha x\alpha a I(x) = x\alpha I(x)\alpha a$;
thus $(a\alpha x + x\alpha a)\alpha I(x)$
$= a\alpha x\alpha I(x) + x\alpha a\alpha I(x)$
$= x\alpha I(x)\alpha a + x\alpha a\alpha I(x)$
$= x\alpha(a\alpha I(x) + I(x)\alpha a) = x\alpha(a\alpha x + x\alpha a$,
since $a\alpha x + x\alpha a = a\alpha I(x) + I(x)\alpha a$. This tells us that $x\alpha(a\alpha x + x\alpha a) \in S_W$, hence must commute with a.
Now $a\alpha x\alpha(a\alpha x + x\alpha a)$
$= x\alpha(a\alpha x + x\alpha a)\alpha a$
$= x\alpha a\alpha a\alpha x\alpha a + x\alpha x\alpha a\alpha a\alpha a$
$= x\alpha a\alpha a\alpha x\alpha a + x\alpha a\alpha a\alpha x\alpha a$
$= x\alpha a\alpha(x\alpha a + a\alpha x)$
$= x\alpha a\alpha(a\alpha x + x\alpha a)$. Then we get that
$a\alpha x\alpha(a\alpha x + x\alpha a) - x\alpha a\alpha(a\alpha x + x\alpha a) = 0$.
So that $(a\alpha x - x\alpha a)\alpha(a\alpha x + x\alpha a) = 0$.
Therefore $(a\alpha x + 2x\alpha a - x\alpha a)\alpha(a\alpha x + x\alpha a) = 0$.
Hence $(a\alpha x + x\alpha a)\alpha(a\alpha x + x\alpha a) = 0$ [since $2x\alpha a = 0$].
Thus if $a \in S_w$, $x \in W$ then
$$(a\alpha x + x\alpha a)\alpha(a\alpha x + x\alpha a) = 0.$$
Suppose that $a \in S_W$ and $a \notin Z(S_W)$, the center of S_W then $b = a\alpha x + x\alpha a \neq 0$ for some $x \in W$. By the above, $b\alpha b = 0$ and $(b\alpha y + y\alpha b)\alpha(b\alpha y + y\alpha b) = 0$ for all $y \in W$.
Then $0 = b\alpha(b\alpha y + y\alpha b)\alpha(b\alpha y + y\alpha b)\alpha y$
$= (b\alpha b\alpha y + b\alpha y\alpha b)\alpha(b\alpha y + y\alpha b)\alpha y$
$= b\alpha y\alpha b\alpha(b\alpha y\alpha y + y\alpha b\alpha y)$ since $b\alpha b = 0$
$= b\alpha y\alpha b\alpha\, b\alpha y\alpha y + b\alpha y\alpha b\alpha\, y\alpha b\alpha y$
$= (b\alpha y)\alpha(b\alpha y)\alpha(b\alpha y)$
Hence $0 = (b\alpha y)^2\alpha(b\alpha y)$.
Thus $(b\alpha y)^2\alpha(b\alpha y) = 0$. Since W is an ideal of a semi-prime Γ-ring M, W is semi-prime. By Lemma 3.4,

$(b\alpha y)^2\alpha(b\alpha y) = 0$ for al $y \in W$ forces $b = 0$.By Lemma 3.12, we get $S_W \subset Z(S_W) \subset Z$.

Let $\widetilde{M} = \frac{M}{W}$; since M is semi-prime, $\widetilde{M}$ is 2-torsion free and semi-prime. Let $I(\tilde{x}) = \tilde{x},\ I(\tilde{y}) = \tilde{y}$ in $\widetilde{M}$ and let x, y be inverse image in M of $\tilde{x},\ \tilde{y}$ respectively. Thus $I(x) - x$ and $I(y) - y$ in W. Hence $2I(x) = 2x, 2I(y) = 2y$.Since S is commutative, we have $(2x)\alpha(2y) = (2y)\alpha(2x)$
So that $4x\alpha y = 4y\alpha x$. Thus $4(x\alpha y - y\alpha x) = 0$. Because M is semi-prime, we have
$2(x\alpha y - y\alpha x) = 0$ which translates $\widetilde{M}$ into $\tilde{x}\alpha\tilde{y} = \tilde{y}\,\alpha\tilde{x}$.
Because $\widetilde{M}$ is semi-prime, 2-torsion free and any two of its symmetric elements commute, all symmetric elements of $\widetilde{M}$ are in $Z(\widetilde{M})$, the centre of $\widetilde{M}$. If $s \in S$ then $(\tilde{s}) = \tilde{s}$, hence we have $s\alpha x - x\alpha s \in W$ for all $x \in M$. But $s\alpha\big(x + I(x)\big) = \big(x + I(x)\big)\alpha s$. Then
$s\alpha x - x\alpha s = I(x)\alpha s - s\alpha I(x) = \mathrm{I}(s\alpha x - x\alpha s)$. Thus
$s\alpha x - x\alpha s \in S$. Hence $s\alpha x - x\alpha s \in S_W \subset Z(W) \subset Z$.
Also $s\alpha(s\alpha x) - (s\alpha x)\alpha s = s\alpha(s\alpha x) - s\alpha(x\alpha s)$

$$= s\alpha(s\alpha x - x\alpha s) .$$

Therefore $(s\alpha(s\alpha x - x\alpha s))\alpha x = x\alpha(s\alpha(s\alpha s - x\,\alpha s))$.
Because $s\alpha x - x\,\alpha s \in Z$ this gives us
$(s\alpha x - x\alpha s)\alpha(s\alpha x - x\alpha s) = 0$. However $s\alpha x - x\alpha s$ is in Z and is nilpotent, which can happen in the semi-prime Γ-ring M only if $s\alpha x - x\alpha s = 0$.Therefore $s\alpha x = x\alpha s$ and $s \in Z$.
Thus $S \subset Z$ has been proved.

The theorem has the following consequence

Theorem 3.15. Let M be a simple Γ-ring with involution. If dim $Z > 4$, then $\bar{S} = M$.

Proof. By theorem 3.14 any non-zero ideal of M is contained in $\bar{S}$ in which case $\bar{S} = M$. Thus the proof is completed.

Theorem 3.16. If M is a semi-prime Γ-ring with involution I in which every symmetric element $s \neq 0$ is invertible in M then M is

(i) a division Γ-ring, or
(ii) The direct sum of a division Γ-ring and its opposite, relative to the exchange involution $\mathrm{I}((x,y)_\alpha) = (y,x)_\alpha\ \ x,y \in M, \alpha \in \Gamma$ or
(iii) The 2×2 matrices over a Γ_2-field relative to the sympletic involution, namely.

$$I\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof. If $A \neq 0$ is an ideal of M such that $I(A) = A$ then A contains no invertible elements, hence A can have no non-zero symmetric elements. However, if $0 \neq x \in A$ then $I(x) \in A$ and $t = x + I(x) \in A$ is symmetric. In consequence $t = 0$ and $I(x) = -x \neq 0$ .Thus A can not have any 2-torsion, otherwise A would contain non-zero symmetric elements. If $x \in A$ and $\alpha \in \Gamma$ then $x\alpha I(x) \in A$ is symmetric. Thus $x\alpha I(x) = 0$. But since $I(x) = -x$, this gives us that $0 = x\alpha I(x) = x\alpha(-x) = -x\alpha x$. So that $x\alpha x = 0$ for all $x \in A$. Since A is 2-torsion free, we get from the fact that all its elements have square 0, that is $(A\Gamma)^2 A = 0$, contradicting the semi-primeness of M. Hence we may assume that M has no proper non-zero ideals with involution. Suppose that $N \neq 0$ is an ideal of M. Then since $N + I(N)$ is a non-zero ideal of M, $N + I(N) = M$.Also $N \cap I(N) \neq M$ is an ideal of M, hence $N \cap I(N) = 0$. Thus $M = N \oplus I(N)$, where $\oplus$ means internal direct sum. We claim that N is a division Γ-ring. For $1 = e + I(f),\ e, f \in N$ gives us that e is the unit element of N. Also, if $x \neq 0$ is in N, then $x + I(x) \neq 0$ is invertible in M; if its inverse is $y_1 + I(y_2)$ where $y_1, y_2 \in N$,we get that $x\alpha y_1 = e$. Thus N is a division Γ-ring as is $I(N)$. Clearly, the involution on M is the exchange involution.

Finally suppose that M is simple but not a division Γ-ring. Let $0 \neq x \in M$ be non-invertible in M. Since $x\Gamma S\Gamma\, \mathrm{I}(x) \subset \mathrm{S}$ and no element of $x\Gamma S\Gamma\, \mathrm{I}(x)$ is invertible, $x\Gamma S\Gamma\, \mathrm{I}(x) = 0$. Thus if $m \in M$, and $\alpha \in \Gamma$ then
$x\alpha\big(m + I(m)\big)\alpha I(x) = 0$, that is,
$x\alpha m\alpha I(x) = -x\alpha I(m)\alpha I(x) = -I(x\alpha m\alpha I(x))$. Since M is simple, $k = x\alpha m\alpha I(x) \neq 0$, then $I(k) = -k \neq 0$ is not invertible, so $k\alpha k = 0$ in M.
In particular char $M \neq 2$, then $k\Gamma S\Gamma \mathrm{k} \subset S$ and consists of non-invertible elements. Hence
$k\Gamma S\Gamma \mathrm{k} = 0$. If $\mathrm{s} \in \mathrm{S}$ then $s\alpha k - k\alpha s \in S$. But

$$\begin{aligned}(k\alpha s - s\alpha k)\alpha(k\alpha s - s\alpha k) &= (k\alpha s)\alpha(k\alpha s) - s\alpha k\alpha k\alpha s - k\alpha s\alpha s\alpha k + (s\alpha k)\alpha(s\alpha k)\\ &= (k\alpha s\alpha k)\alpha s - s\alpha(k\alpha k)\alpha s - (k\alpha s\alpha k)\alpha k^{-1}\alpha s\alpha k + s\alpha(k\alpha s\alpha k)\end{aligned}$$

$= 0$ since $k\Gamma S\Gamma \mathrm{k} = 0$ and $k\alpha k = 0$. Therefore $k\alpha s - s\alpha k = 0$. Hence $k\alpha s = s\alpha k$ Thus k centralizer S, hence $\bar{S}$. Since $k \notin Z$ being nilpotent, $\bar{S} = M$. By Theorem 3.15, we get that dim $Z = 4$. Since M is not a division Γ-ring, M is Γ-isomorphic with F_2 , where F_2 be the Γ_2-field of set of all 2×2 matrices whose entries comes from a Γ-field F and Γ_2 be the set of all 2×2 matrices whose entrices comes from Γ. From the nature of the involution on F_2 it is easy to see that the involution I on F_2 must be the symplectic involution. This finishes the proof.

Theorem 3.17. Let M be a simple Γ-ring with involution I. If dim $\mathrm{Z} > 4$, then $\bar{K}_0 = M$.

Proof. If char $M = 2$, then $\mathrm{K}_0 = T$ and so $\bar{K}_0$ is a Lie ideal of M. As such, since dim $\mathrm{Z} > 4$, by Theorem 3.8 we have that either $\bar{K}_0 \subset Z$ or $\bar{K}_0 \supset [M, M]_\Gamma$. In this latter case, $\bar{K}_0$ contains the sub-Γ-ring which is generated by $[M, M]_\Gamma$, which is easily seen to be M. So we may suppose that $\bar{K}_0$ and so K is contained in Z.

If $x \in M$ then $x + I(x) \in K_0 \subset Z$. Since $K_0 \neq 0$, there is a $u \in M$ such that $p = u + I(u) \neq 0$ is in Z. Now $p\alpha\, I(x)\alpha x = I(x)\alpha\, p\alpha x = I(x)\alpha(u + I(u))\alpha x$, so is in $K_0 \subset Z$; in consequence $I(x)\alpha x \in Z$. Thus x satisfies $x\alpha x - \big(x + I(x)\big)\alpha x - I(x)\alpha x = 0$, a quadratic equation over Z for every $x \in M$ .This gives the contradiction that dim $\mathrm{Z} \leq 4$. So if char $M = 2$ we are done. Now we may assume that char $M \neq 2$ and so $K_0 = K$. Since K Γ K is a Lie ideal of M, the sub-Γ-ring it generates, $\overline{K\,\Gamma\,K}$, is a sub- Γ-ring and also a Lie ideal of M. By theorem 3.7, $\overline{K\,\Gamma\,K} = M$ or K Γ K ⊂ Z. So if $\overline{K\,\Gamma\,K} \neq M$ we must have that K Γ K ⊂ Z. Our objective will be to show that dim $\mathrm{Z} \leq 4$ results from this.

Given $a, b \in \mathrm{K}$ then $a\alpha b \in$ K Γ K ⊂ Z. Hence since Z is a Γ-field, $a\alpha b = 0$ or a must be invertible in M. If a is not invertible, we get $a\,\Gamma\,\mathrm{K} = 0$. If $s \in S$ then $s\alpha a\alpha s \in K$. Hence $a\alpha s\alpha a\alpha s = 0$. Given $x \in M,\ x = s + k$ where $s = \frac{x+I(x)}{2} \in S,\ k = \frac{x-I(x)}{2} \in K$.

Then $((a\alpha x)\alpha)^2(a\alpha x)$
$= (a\alpha x)\alpha(a\alpha x)\alpha(a\alpha x)$
$= a\alpha(s+k)\alpha(a\alpha(s+k)\alpha a\alpha(s+k)$
$= (a\alpha s + a\alpha k)\alpha a\alpha(s+k)\alpha a\alpha(s+k)$
$= (a\alpha s\alpha a + a\alpha k\alpha a)\alpha(s+k)\alpha a\alpha(s+k)$
$= \{(a\alpha s\alpha a(s+k) + a\alpha k\alpha a\alpha(s+k)\}\alpha a\alpha(s+k)$
$= (a\alpha s\alpha a\alpha s + a\alpha s\alpha a\alpha k + a\alpha k\alpha a\alpha s + a\alpha k\alpha a\alpha k)\alpha a\alpha(s+k)$
$= (0 + a\alpha s\alpha 0 + 0\alpha a\alpha s + 0)\alpha a\alpha(s+k)$, since $a\alpha k = 0$
$= 0$
This shows that every element in $a\Gamma M$ has cube 0. But this violetes the fact that M is simple, hence semi-prime (by lemma 3.4) unless $a = 0$. In short, $a \neq 0 \in K$ implies that a is invertible in M.

If $0 \neq a \in K$ then $a\alpha a = q \neq 0 \in Z$ whence $a^{-1} = q^{-1}\alpha a$. Since $a\Gamma K \subset Z$, we have that $K \subset Z\Gamma a^{-1} = Z\Gamma a$. If $s \in S$ commutes with a, since $s\alpha a$ must then be in K. $s\alpha a = p\alpha a$ for some $p \in Z$ leading us to $s = p \in Z$. In other words, the only symmetric commuting with a are those in the centre.

Given $s \in S, s\alpha a + a\alpha s \in K$, hence $s\alpha a + a\alpha s = q_1\alpha a, q_1 \in ZIS$. Thus $\left(s - \frac{q_1}{2}\right)\alpha a + a\alpha\left(s - \frac{q_1}{2}\right) = 0$. But then $\left(s - \frac{q_1}{2}\right)\alpha\left(s - \frac{q_1}{2}\right)$ comumutes with a; being symmetric, $\left(s - \frac{q_1}{2}\right)\alpha\left(s - \frac{q_1}{2}\right)$ must be in Z.

Given $x \in M$ then $x = s + k$ where $s \in S$ and $k \in K$. Hence $x = s + \lambda\alpha a$ where $\lambda \in Z$. Since $s\alpha a + a\alpha s = \mu\alpha a, \mu \in Z$,we have as above that $\left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right) \in Z$. Then $\left(x - \frac{\mu}{2}\right)\alpha\left(x - \frac{\mu}{2}\right)$
$= \left(s + \lambda\alpha a - \frac{\mu}{2}\right)\alpha\left(s + \lambda\alpha a - \frac{\mu}{2}\right)$
$= \left\{\left(s - \frac{\mu}{2}\right) + \lambda\alpha a\right\}\alpha\left\{\left(s - \frac{\mu}{2}\right) + \lambda\alpha a\right\}$
$= \left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right) + \lambda\alpha a\left(s - \frac{\mu}{2}\right) + \lambda\alpha a\alpha\lambda\alpha a + \left(s - \frac{\mu}{2}\right)\alpha\lambda\alpha a$
$= \left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right) + \lambda\alpha a\alpha\left(s - \frac{\mu}{2}\right) + \lambda\alpha\left(s - \frac{\mu}{2}\right)\alpha a + \lambda\alpha\lambda\alpha a\alpha a$, since $\lambda \in Z$
$= \left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right) + \lambda\alpha(a\alpha\left(s - \frac{\mu}{2}\right) + \left(s - \frac{\mu}{2}\right)\alpha a) + \lambda\alpha\lambda\alpha a\alpha a$
$= \left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right) + \lambda\alpha\lambda\alpha a\alpha a$ since $\left(a\alpha\left(s - \frac{\mu}{2}\right) + \left(s - \frac{\mu}{2}\right)\alpha a\right) = 0.$
$= \left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right) + \lambda\alpha\lambda\alpha a\alpha a.$
Hence $\left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right) + \lambda\alpha\lambda\alpha a\alpha a$ is in Z, since both $a\alpha a$ and $\left(s - \frac{\mu}{2}\right)\alpha\left(s - \frac{\mu}{2}\right)$ are in Z. Therefore M is quadratic over Z, in consequence of which dim ≤ 4 , a contradiction. Thus the proof is finished

Theorem 3.18. Let M be a Γ-ring with involution in which $2M = M$. If $\overline{K} = M$ then $S = (K,K)_\Gamma$ Hence $M = K + (K,K)_\Gamma$. In other words, given $s \in S$ then $s = \sum k_i\alpha k_i - \sum q_j\alpha q_j$, where $k_i \in K$, $q_j \in K$ and $\alpha \in \Gamma$.
Proof. If $a, b \in K$ then $a\alpha b - b\alpha a \in K$, $a\alpha b + b\alpha a \in (K,K)_\Gamma$ and $2a\alpha b = (a\alpha b - b\alpha a) + (a\alpha b - b\alpha a) \in K + (K,K)_\Gamma$. Since $2M = M$ we get from this that $K\Gamma K \subset K + (K,K)_\Gamma$.

We claim that $(K,K)_\Gamma\Gamma K \subset K + (K,K)_\Gamma$. For let $a, c \in K$; then $(a\alpha a)\alpha c + c\alpha(a\alpha a) \in K$, and
$(a\alpha a)\alpha c - c\alpha(a\alpha a) = a\alpha(a\alpha c - c\alpha a) + (a\alpha c - c\alpha a)\alpha a \in (K,K)_\Gamma$.
Hence $2(a\alpha a)\alpha c =$
$(a\alpha a)\alpha c + c\alpha(a\alpha a) + (a\alpha a)\alpha c - c\alpha(a\alpha a) \in K + (K,K)_\Gamma$
and so $(a\alpha a)\Gamma K \subset K + (K,K)_\Gamma$. Linearizing this on a getting thereby $(K,K)_\Gamma\Gamma K \subset K + (K,K)_\Gamma$.
Now $K\Gamma K \subset K + (K,K)_\Gamma$ hence
$(K\Gamma)^2K = (K\Gamma K)\Gamma K \subset K\Gamma K + (K,K)_\Gamma\Gamma K \subset K + (K,K)_\Gamma$.
Continuing, we get $(K\Gamma)^nK \subset K + (K,K)_\Gamma$ for all n. But $\overline{K} = \sum(K\Gamma)^nK$ and since
$\overline{K} = M$, $M = \sum(K\Gamma)^nK \subset K + (K,K)_\Gamma$. Thus
$M = K + (K,K)_\Gamma$. Since $2M = M$, we get from this
$S = (K,K)_\Gamma$. Thus the theorem is proved.

Now we close this section with a discussion of the ideal structure of S as a Jordan Γ-ring in semi-prime Γ-rings.

The result is

Theorem 3.19. Suppose that M is a semi-prime, 2-torsion free Γ-ring with involution I. Let $U \neq 0$ be a Jordan ideal of S. Then there exists an ideal $W \neq 0$ of M such that $U \supset \{w + I(w) | w \in W\}$.

Proof. If $a \in U, s \in S$ then since
$2a\alpha a = a\alpha a + a\alpha a \in U$, $4s\alpha a\alpha s =$
$2\{(a\alpha s + s\alpha a)\alpha a + a\alpha(a\alpha s + s\alpha a)\} - 2(a\alpha a\alpha s + s\alpha a\alpha a) \in U$.
Similarly we show that $4a\alpha s\alpha a \in U$. If $x \in M$ then
$4(a\alpha a\alpha x + I(x)\alpha a\alpha a)$
$= 4\{a\alpha(a\alpha x + I(x)\alpha a) + (a\alpha x + I(x)\alpha a)\alpha a\}$
$-4a\alpha(x + I(x))\alpha a$,
so by the above is in U. That is, $4(a\alpha a\alpha x + I(x)\alpha a\alpha a) \in U$ for all $a \in U, x \in M$.
Hence $2\{4(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha 4(a\alpha a\alpha x + I(x)\alpha a\alpha a)\} \in U$.
Expanding this we have
$2\{4(a\alpha a\alpha x + I(x)\alpha a\alpha a)\}\alpha 4(a\alpha a\alpha x + I(x)\alpha a\alpha a)$
$= 8(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha 4(a\alpha a\alpha x + I(x)\alpha a\alpha a)$
$= 32(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)$
$=$
$32(a\alpha a\alpha x\alpha a\alpha a\alpha a\alpha x + I(x)\alpha a\alpha a\alpha a\alpha a\alpha a\alpha x + a\alpha a\alpha x\alpha I(x)\alpha a\alpha a + I(x)\alpha a\alpha a\alpha I(x)\alpha a\alpha a)$
$= 32(a\alpha a\alpha x\alpha a\alpha a\alpha a\alpha x + I(x)\alpha a\alpha a\alpha I(x)\alpha a\alpha a) + 32a\alpha a\alpha x\alpha I(x)\alpha a\alpha a + 32I(x)\alpha(a\alpha)^3a\alpha x.$
$= 32\{(a\alpha a\alpha x\alpha a\alpha a\alpha a\alpha x + I(a\alpha a\alpha x\alpha a\alpha a\alpha a\alpha x)$
$+32a\alpha a\alpha x\alpha I(x)\alpha a\alpha a + 32I(x)\alpha(a\alpha)^3a\alpha x.$
By our results above, since $2a\alpha a \in U$ we have
$32a\alpha a\alpha x\alpha I(x)\alpha a\alpha a \in U$ and
$32\{(a\alpha a\alpha x\alpha a\alpha a\alpha a\alpha x + I(a\alpha a\alpha x\alpha a\alpha a\alpha a\alpha x)\} \in U$; the net result of all this is that
$32I(x)\alpha(a\alpha)^3a\alpha x \in U$ for all $x \in M, a \in U$. Linearize this on x; this yields
$32(I(x)\alpha(a\alpha)^3a\alpha y + I(y)\alpha(a\alpha)^3a\alpha x) \in U$. Let
$W = 32M\Gamma(a\alpha)^3a\Gamma M$; W is an ideal of M and moreover, if We see some consequences of this ge
$w \in W$ then $w + I(w) \in U$ follows from
$32(I(x)\alpha(a\alpha)^3a\alpha y + I(y)\alpha(a\alpha)^3a\alpha x)$
$= 32\{I(x)\alpha(a\alpha)^3a\alpha y + I(I(x)\alpha(a\alpha)^3a\alpha x)\}$ being in U. So W does the trick, provided $W \neq 0$. If $W = 0$, then $M\Gamma(a\alpha)^3a\Gamma M = 0$ for all $a \in U$. Since M is 2- torsion free semi-prime Γ-ring, this would force $(a\alpha)^3a = 0$ for all $a \in U$. Now for $a \in U$, $2a\alpha a \in U$, then for any $x \in M$, $4(a\alpha a\alpha x + I(x)\alpha a\alpha a) \in U$ as we saw earliar. Hence

$\{(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha\}^3(a\alpha a\alpha x + I(x)\alpha a\alpha a) = 0.$
Multiply this from the right by $a\alpha a$ and from the left by x we get $x\alpha\{(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha\}^3(a\alpha a\alpha x + I(x)\alpha a\alpha a)a\alpha a = x\alpha 0 a\alpha a$

$$\Rightarrow x\alpha\{(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha\}^3(a\alpha a\alpha x a\alpha a + I(x)\alpha a\alpha a\alpha a\alpha a) = 0$$

$$\Rightarrow x\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x a\alpha a + I(x)\alpha(a\alpha)^3 a) = 0$$

$$\Rightarrow x\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x a\alpha a),$$

since $(a\alpha)^3 a = 0.$

$$\Rightarrow x\alpha(a\alpha a\alpha x a\alpha a\alpha a\alpha x + I(x)\alpha a\alpha a\alpha I(x)\alpha a\alpha a + a\alpha a\alpha x\alpha I(x)\alpha a\alpha a + I(x)\alpha(a\alpha)^3 a\alpha x)\alpha(a\alpha a\alpha x + I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x a\alpha a) = 0.$$

$$\Rightarrow x\alpha(a\alpha a\alpha x a\alpha a\alpha a\alpha x + I(x)\alpha a\alpha a\alpha I(x)\alpha a\alpha a + a\alpha a\alpha x\alpha I(x)\alpha a\alpha a)\alpha(a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a + I(x)\alpha a\alpha a\alpha a\alpha a\alpha a\alpha x a\alpha a) = 0$$

$$\Rightarrow x\alpha(a\alpha a\alpha x a\alpha a\alpha a\alpha x + I(x)\alpha a\alpha a\alpha I(x)\alpha a\alpha a + a\alpha a\alpha x\alpha I(x)\alpha a\alpha a)\alpha a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a = 0$$

$$\Rightarrow x\alpha(a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a + I(x)\alpha a\alpha a\alpha I(x)\alpha a\alpha a\alpha a\alpha a\alpha x a\alpha a\alpha x\alpha x a\alpha a\alpha x + a\alpha a\alpha x\alpha I(x)\alpha a\alpha a\alpha a\alpha a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a) = 0$$

$$\Rightarrow x a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a\alpha a\alpha x a\alpha a + x\alpha I(x)\alpha a\alpha a\alpha I(x)\alpha(a\alpha)^3 a\alpha x a\alpha a\alpha a\alpha x a\alpha a + x a\alpha a\alpha x\alpha I(x)\alpha(a\alpha)^3 a\alpha x a\alpha a\alpha a\alpha x a\alpha a = 0$$

$$\Rightarrow (x a\alpha a)\alpha(x a\alpha a)\alpha(x a\alpha a)\alpha(x a\alpha a)\alpha(x a\alpha a) + 0 = 0.$$

Hence $\{(x a\alpha a)\alpha\}^4(x a\alpha a) = 0$ for all $x \in M$. However, this forces the existence of a non-zero nilpotent ideal in M, unless $a\alpha a = 0$ for all $a \in U$. In consequence, $a\alpha a = 0$ for all $a \in U$. Linearizing this on a; we get $a\alpha b + b\alpha a = 0$ for all $a, b \in U$. If $s \in S$, $b = a\alpha s + s\alpha a \in U$, hence
$0 = a\alpha b + b\alpha a = a\alpha(a\alpha s + s\alpha a) + (a\alpha s + s\alpha a)\alpha a = a\alpha a\alpha s + a\alpha s\alpha a + a\alpha s\alpha a + s\alpha a\alpha a$
$= 2a\alpha s\alpha a$, since $a\alpha a = 0$. But M is 2-torsion free, hence $a\Gamma S\Gamma a = 0$ for all $a \in U$. If $k \in K$ then $k\alpha a\alpha k \in S$, therefore, $a\alpha k\alpha a\alpha k\alpha a = 0$. If $x \in M$, then $2x = s + k$ where $s = x + I(x)$, $k = x - I(x)$.
Then $\{(2a\alpha x)\alpha\}^2(2a\alpha x)$
$= 2a\alpha x\alpha 2a\alpha x\alpha 2a\alpha x$
$= (a\alpha 2x)\alpha(a\alpha 2x)\alpha(a\alpha 2x)$
$= (a\alpha(s + k)\alpha a\alpha(s + k)\alpha a\alpha(s + k)$
$= (a\alpha s + a\alpha k)\alpha a\alpha(s + k)\alpha a\alpha(s + k)$
$= (a\alpha s\alpha a + a\alpha k\alpha a)\alpha(s + k)\alpha(a\alpha s + a\alpha k)$
$= a\alpha k\alpha a\alpha(s + k)\alpha(a\alpha s + a\alpha k)$ since $a\alpha s\alpha a = 0$
$= a\alpha k\alpha a\alpha s + a\alpha k\alpha a\alpha k)\alpha(a\alpha s + a\alpha k)$
$= a\alpha k\alpha a\alpha s\alpha a\alpha s + a\alpha k\alpha a\alpha k\alpha a\alpha s + a\alpha k\alpha a\alpha s\alpha a\alpha k + a\alpha k\alpha a\alpha k\alpha a\alpha k$
$= a\alpha k\alpha(a\alpha s\alpha a)\alpha s + (a\alpha k\alpha a\alpha k\alpha a)\alpha s + a\alpha k\alpha(a\alpha s\alpha a)\alpha k + (a\alpha k\alpha a\alpha k\alpha a)\alpha k$
$= a\alpha k\alpha 0 + 0\alpha s + a\alpha k\alpha 0 + 0\alpha k$
$= 0.$
So that $\{(a\alpha x)\alpha\}^2(a\alpha x) = 0$. This is not possible in a semi-prime Γ-ring. In short $a = 0$ follows. Since $U \neq 0$, we must have $W = 32M\Gamma(a\alpha)^3 a\Gamma M \neq 0$ for some $a \in U$. As we have seen $w + I(w) \in U$ for all $w \in W$.Note that $\{w + I(w) | w \in W\} \neq 0$, otherwise $w = -I(w)$ for all $w \in W$; this leads $w\alpha w = 0$ and so $(W\Gamma)^2 W = 0$ in the 2-torsion free Γ-ring. Because $W \neq 0$ and M is semi-prime, this too is not possible. Thus the proof is finished.

Conclusion

In this manuscript, some characterizations of Lie and Jordan ideals of simple and semi-prime gamma rings with involution are studied. The concept of Lie and Jordan structures of gamma rings is applied to proving some Lemmas and Theorems. It would play an important role in the research field of mathematics.

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