International Journal of Advanced Research and Publications
ISSN: 2456-9992

Chromatic Index Of Some Classes Of Graph

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Abstract: Graphs considered in this study are finite simple graphs. Edge coloring properties of fan graph, star graph, bistar graph, helm graph, and friendship graph are investigated and their chromatic indices are determined. The chromatic index determines the minimum number of color used to color the edges of a particular graph. Also, the study aimed to determine the chromatic index polynomial of the graphs stated above. Vizing's classifications of graph based on its maximum degree are also taken into consideration. This study, as a pure research, is descriptive and expository in nature. This provided comprehensive description of the concepts in mathematics, particularly on the context of graph theory, which may create an abstraction of the concepts of some classes of graphs and the theory of edge coloring. Results of the study showed that almost all graphs stated above are of class 1. Their chromatic indices were determined based on their maximum degree. Furthermore, the chromatic index polynomial of the graphs considered in this study is obtained to determine the number of ways to color the edges of a particular graph by using at most λ colors

Keywords: Chromatic Index, Chromatic Index Polynomial

1. Introduction

Edge coloring has been an interesting subject in graph theory. From the problem of coloring a geographical map by using only four colors, the theory of graph coloring has originated and coloring the edges of a particular graph was introduced. Here, the chromatic index denoted by λ', described the minimum number of colors to color the edges of a particular graph such that no adjacent edges were assigned with similar colors. Vizing's classifications of graph based on its maximum degree, Δ, are also taken into consideration [1]. A graph, G is classified as class 1 graph if $\lambda'(G) = \Delta(G)$ and it is of class 2 if $\lambda'(G) = \Delta(G)+1$. Since its discovery, this research interest has grown into numerous studies. For instance, Beseri [2], investigated the old and new results on the classical edge coloring as well as the generalized edge coloring problems. In this paper, the author developed some algorithms and modules by using Combinatorica package to color the edges of graphs with Mathematica, a new web – based technology. A strong edge coloring of a graph was studied in [3]. This parameter of graph coloring is described as a proper edge coloring in which every color class is an induced matching. Here, a strong chromatic index was determined for graph, G, with maximum degree of at most four and maximum average degree of less than 3. Result of the study also showed that the list strong chromatic index of a graph is, at most, $3\Delta+1$, where Δ is the maximum degree of graph G. Also, if G is a planar graph with maximum degree of at least 4 and girth of at least 7, then the list chromatic index is, at most, 3Δ. Also, in [4], an odd edge coloring was introduced as a characteristic of a proper edge coloring. Here, an edge coloring of a graph G is considered. An edge coloring is said to be odd edge coloring if for every vertex of G, the vertex v uses the color c odd number of times or does not use it at all. Proponents of this study proved that 6 is the minimum number of colors that can be used for an odd edge coloring of any loopless connected graph and showed that its upper bound is sharp. Motivated by the related studies reviewed and aforementioned, this study attempted to investigate the parameters of edge coloring of some classes of graph namely, fan graph, star graph, bistar graph, helm graph and friendship graph. Also, chromatic index polynomial is investigated in this study. In addition, the study also investigated common classes of graph and identified its class either as class 1 or class 2 graph as presented by Vizing.

2. Preliminaries

Definition 2.1.[5]A graph G is an ordered pair $(V(G), E(G))$, where $V(G)$ a nonempty set of elements is called vertices and $E(G)$ is a set of an ordered pairs of vertices called edges.

A graph, G is characterized by its order (number of vertices) and size (number of edges).

Definition 2.2.[5] Let G be graph and let $v_i \in V(G)$. The degree of v_i, denoted by $\deg(v_i)$ is the number of edges incident with v_i.

Definition 2.22. [5] An edge coloring of graph G is a function $f : E(G) \to C$ where $E(G)$ is the edge set of G and C is the set of distinct colors.

Theorem 2.1. [6] A graph G is $k-$ edge colorable if its edges can be colored with $k-$ colors so that no two adjacent edges have the same color and $\chi'(G) \le k$.

Theorem 2.2. [6] The line – chromatic number of graph G satisfies the inequalities $\Delta(G) \le \chi'(G) \le \Delta(G)+1$.

This theorem is due to Vizing [7] and [1]. This means that the chromatic index of a graph G is relative to its maximum degree. Also from this theorem, graphs were classified into two classes:

Class 1: $\chi'(G) = \Delta(G)$

Class 2: $\chi'(G) = \Delta(G)+1$

Theorem 2.3. [7] Let G_n be set of graphs of order n and G'_n, set of graphs of order n and of class 1. Then the number of graphs of order n and of class 1, $|G'_n|$ is given by

$$\lim_{n\to\infty} \frac{|G'_n|}{G_n} = 1.$$

Theorem 2.3 indicates that almost all graphs are of class 1.

3. Main Results

This section presents the chromatic index and chromatic index polynomials of some classes of graph.

3.1. Chromatic Index of Some Classes of Graph

Coloring properties of special classes of graphs such as fan, star, bistar, helm, and friendship graphs are considered in this section and their chromatic indices and chromatic index polynomials are presented in this section.

3.1.1. Fan Graph F_n

A fan graph F_n is formed by joining all vertices of a path graph, P_n to another vertex called the center. Thus, F_n has $n+1$ vertices (order) and $2n-1$ edges (size). Figure 1 shows the pictorial representation of a fan graph.

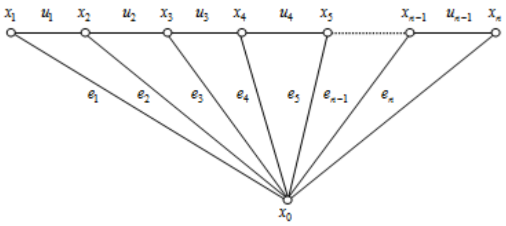

Figure 1. Labeled Fan Graph, F_n

The labeling of fan graph presented in figure 1 is considered and the next theorem gives its chromatic index.

Theorem 3.1. Let n be positive integer and $n \geq 3$. The chromatic index of a fan graph of order $n+1$, F_n is $\chi'(F_n) = n$.

Proof: Let $E = \{e_1, e_2, e_3, ..., e_{n-1}, e_n, u_1, u_2, u_3, ..., u_{n-1}\}$ and $V = \{x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n\}$ be the sequence of edges and vertices of a fan graph of order $n+1$. Also, let $C : E(F_n) \rightarrow N$ be the edge coloring of F_n defined by

$$c(e_i) = i \text{ and } c(u_i) = \begin{cases} 3 & if \quad i = 1 \\ 2 & if \quad i \text{ is } odd \\ 1 & if \quad i \text{ is } even \end{cases}.$$

Note that C is a proper edge coloring of F_n, thus F_n is an $n-edge$ colorable. By Theorem 2.1 $\chi'(F_n) \leq n$. It is now easy to show that the edges $e_1, e_2, e_3, ..., e_{n-1}, e_n$ can be colored with $n-$ colors. Also, the sequence of edges $u_1, u_2, u_3, ..., u_{n-1}$ can be colored using $c(u_i)$ such that $c(u_i)$ is absent to the edges $\{e_n, e_{n+1}\}$ or at least $n-$ colors are required to color F_n Thus, $\chi'(F_n) \geq n$. Therefore, F_n is an $n-edge$ colorable and $\chi'(F_n) = n$ as desired. $\square$

The following remark is an obvious observation from the above illustration and from theorem 3.1.

Remark 3.1. A fan graph of order $n+1$, F_n, is a class 1 graph. Thus, $\chi'(F_n) = \Delta(F_n)$.

An edge independence number of a non – empty graph G is the maximum number of edges in an independent set of edges of G. If we let E_i be the independent set of edges of a fan graph F_n, and if n is odd, then we have

$$E_1 = \{e_1, u_2, u_4, u_6, ..., u_{n-1}\}$$
$$E_2 = \{e_2, u_3, u_5, u_7, ..., u_n\}$$
$$E_3 = \{e_3, u_1\}$$
$$E_4 = \{e_4\}$$
$$E_5 = \{e_5\}$$
$$\vdots$$
$$E_{n-1} = \{e_{n-1}\}$$
$$E_n = \{e_n\}$$

These independent sets E_i are called edge color classes and each E_i is assigned with $i-$color. This supports Theorem 3.1 that F_n is an $n-edge$ colorable, since $|E| = |C| = n$. If n is odd, we can say that $\alpha'(F_n) = \frac{n+1}{2}$ and $\alpha'(F_n) = \frac{n}{2}$ if n is even. The following remark shows the relationship of the chromatic index and the edge independence number of F_n.

Remark 3.2. $\chi'(F_n) > \alpha'(F_n)$

3.1.2. Star Graph, S_n

A star graph S_n consists of an independent set of $n-$ vertices each of which is adjacent to a common vertex called the center. The star graph has an order $n+1$ and size n and is isomorphic to a complete bipartite graph, $K_{1,n}$ [8]. Its chromatic number is 1 if $n=1$ and $\chi(S_n) = 2$ if $n \geq 2$.

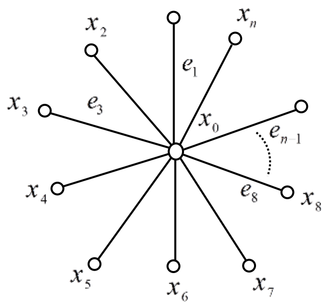

Figure 2. Star graph, S_n, of order $n+1$

Theorem 3.2. The star graph S_n is a class 1 graph, thus, $\chi'(S_n) = \Delta(S_n)$.

Proof: Without loss of generality, it is easy to show that the maximum degree of S_n is n since by definition, each independent vertices of S_n is incident to the central vertex. If $e_1, e_2, e_3, ..., e_{n-1}, e_n$ are the sequence of edges of S_n, then we assign a color mapping $C : E(S_n) \rightarrow N$ such that $c(e_i) = i$, where $i = n$. Thus, $\chi'(S_n) = n$. And since $\Delta(S_n) = n$, then $\chi'(S_n) = \Delta(S_n)$. $\square$

It can also be observed from Figure 2 that $\alpha'(S_n) = 1$ since if S_n is partitioned into E_i edge color classes, $|E_i| = 1$. Also, since a star graph of order $n+1$ is isomorphic to a complete bipartite graph, $K_{1,n}$, then $\chi'(S_n) = \chi'(K_{1,n})$

3.1.3. Bistar Graph, $B_{m,n}$

The bistar graph denoted by $B_{m,n}$, is a graph obtained by joining star graphs S_m and S_n and the vertices of a path graph of order 2. The order and size of a bistar graph is $|V| = m + n + 2$ and $|E| = m + n + 1$, respectively. Figure 3 shows a bistar graph, $B_{m,n}$. The chromatic index of a star graph presented in Theorem 3.3 will be used to determine the proper edge coloring of bistar graph.

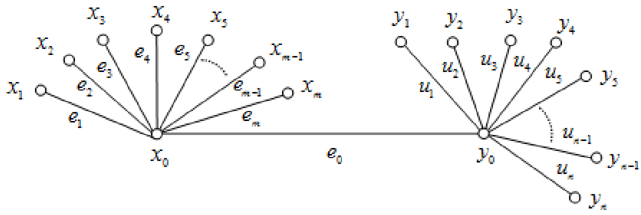

Figure 3. Labeled bistar graph, $B_{m,n}$

Theorem 3.4. The chromatic index of a bistar graph, $B_{m,n}$ is

$$\chi'(B_{m,n}) = \begin{cases} m+1 & \text{if } m \geq n \\ n+1 & \text{if } m < n \end{cases}$$

Remark 3.3. A bistar graph is a class 1 graph, that is $\chi'(B_{m,n}) = \Delta(B_{m,n})$.

3.1.4. Helm Graph, H_n

A helm graph, H_n is obtained by joining a pendant edge at each vertex of a wheel graph, W_n of order $n+1$. The helm graph is perfect only for $n=3$ [9]. In general, a helm graph, H_n has $2n+1$ vertices and $3n$ edges. A labeled helm graph is presented in Figure 4 and considered in the next theorem.

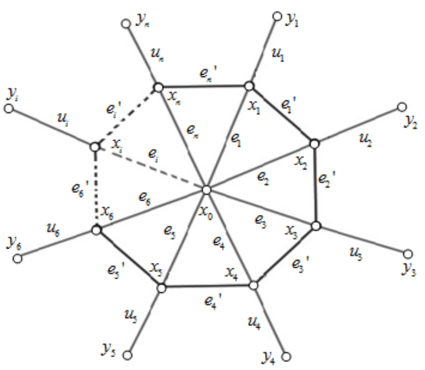

Figure 4. Labeling of Helm Graph, H_n

Theorem 3.6. Let n be positive integers and $n \geq 3$. A helm graph H_n is a class 1 graph, $\chi'(H_n) = \Delta(H_n)$.

3.1.5. Friendship Graph, D_3^n

A friendship graph denoted by D_3^n is a graph obtained by creating n – copies of a cycle graph, C_3 attached to a common vertex. It is also called the Dutch windmill graph [10]. Figure 5 shows a labeled friendship graph.

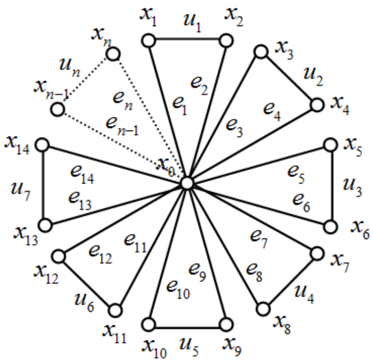

Figure 5. Labeled Friendship Graph

Observe that D_3^n has $2n+1$ vertices (order) and $3n$ edges (size). Specifically, D_3^2 has 5 vertices and 6 edges; D_3^8 has 17 vertices and 24 edges. The next figure shows the labeling of a friendship graph.

Theorem 3.7. Let n be positive integers and $n \geq 2$, $\chi'(D_3^n) = 2n$

Remark 3.6. A friendship graph $\chi'(D_3^n)$ is a class 1 graph, $\chi'(D_3^n) = \Delta(D_3^n)$.

3.2. Chromatic Index Polynomial

The previous section showed the minimum number of color used to color the edges of a graph and the chromatic index of each classes of graph is identified. Obviously, coloring the edges of a graph is not unique. By using at most λ – colors, the chromatic index polynomial gives the number of ways on how to edge color a particular graph.

Definition 3.1. Let λ be the number of available colors to edge – color a graph, G. The chromatic index polynomial denoted by $f(G,\lambda)$ determines the number of ways to color the edges of G with at most λ – colors.

For graphs presented in section 3.1, their chromatic index polynomials are given on the following theorems.

Theorem 3.8. For fan graph, F_n, for $n \geq 3$, $f(F_n,\lambda) = \lambda!(\lambda-1)^{n-2}$.

Proof: Let $V = \{x_0, x_1, x_2, x_3, ..., x_{n-1}, x_n\}$ and $E = \{e_1, e_2, e_3, ..., e_{n-1}, e_n, u_1, u_2, u_3, ..., u_{n-1}\}$ be the sequence of vertices and edges of a fan graph of order $n+1$ as presented in Figure 1. Theorem 3.1 states that F_n is an $n-edge$ colorable and $\chi'(F_n) = n$. Also, let λ be the minimum number of colors used to edge color F_n. Here, it can easily be seen that e_1 can be colored in λ ways. e_2 can be colored in $\lambda - 1$ ways, e_3 in $\lambda - 2$, e_4 in $\lambda - 3$ ways. The pattern of coloring goes until e_n which can be colored in $\lambda - (n-1)$. For the set of edges $u_1, u_2, u_3, ..., u_{n-1}$, we have; u_1 can be colored in $\lambda - 2$ ways since it is adjacent to e_1 and e_2.

Since u_2 is adjacent to e_2 and u_1 and e_3, which may be assigned with similar color, then u_2 can be colored in $\lambda-2$ ways. An obvious observation is that the sequence of edges $u_3,u_4,\ldots,u_{n-2},u_{n-1}$ can also be colored in $\lambda-2$ ways. Hence, $f(F_n,\lambda)=\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-(n-1))(\lambda-2)^{n-2}$. Simplifying the falling factorial, we have $f(F_n,\lambda)=\lambda!(\lambda-2)^{n-2}$ as desired. □

Theorem 3.9. The chromatic index polynomial of a star graph, S_n, for $n\geq 2$, $f(S_n,\lambda)=\lambda!$.
Proof: For $n\geq 2$, it is easy to show that $f(S_n,\lambda)=\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-(n-1))$ since each edge of S_n is attached to a common vertex, x_0. Observe that $\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-(n-1))$ is equal to $\lambda!$. So, $f(S_n,\lambda)=\lambda!$ as desired. □

The number of ways to color the edges of a bistar graph is given by the next theorem.

Theorem 3.10. The chromatic index polynomial of a bistar graph, $B_{n,n}$, for $n\geq 2$, $f(B_{n,n},\lambda)=\lambda!(\lambda-1)!$.

Proof: Consider the sequence of vertices and edges of a bistar graph in Figure 3. Theorem 3.4 states that $B_{n,n}$ is $n+1$ colorable which gives $\lambda=n+1$. The sequence of edges $e_0,e_1,e_2,\ldots,e_{n-1},e_n$ can be colored in $\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n)$ ways. Since the edge set $\{u_1,u_2,\ldots,u_{n-1},u_n\}$ is independent to the edge set $\{e_1,e_2,\ldots,e_{n-1},e_n\}$, then $\{u_1,u_2,\ldots,u_{n-1},u_n\}$ can be colored in $(\lambda-1)(\lambda-2)\ldots(\lambda-n)$ ways. Hence,
$f(B_{n,n},\lambda)=\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-n)(\lambda-1)(\lambda-2)\ldots(\lambda-n)$.
Simplifying, we have $f(B_{n,n},\lambda)=\lambda!(\lambda-1)!$. □

Theorem 3.11. For $n\geq 2$, $f(D_3^n,\lambda)=\lambda!(\lambda-2)^n$
Proof: By theorem 3.7, the chromatic index of a friendship graph $\chi'(D_3^n)=2n$ so $\lambda=2n$. It can easily be seen that the sequence of edges $e_1,e_2,e_3,\ldots,e_{2n-1},e_{2n}$ can be colored in $\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-(2n-1))$ which can be expressed in $\lambda!$ ways. Since u_n is adjacent to e_{2n} and e_{2n-1}, u_n can be colored in $\lambda-2$ ways. The sequence of edges $u_1,u_2,u_3,\ldots,u_{n-1},u_n$ can be colored in $(\lambda-2)^n$. Thus, $f(D_3^n,\lambda)=\lambda!(\lambda-2)^n$. □

Theorem 3.12. For $n\geq 3$, $f(H_n,\lambda)=\lambda!(\lambda-2)(\lambda-3)^{2n-1}$
Proof: It can easily be observed from Figure 4 that the sequence of edges $e_1,e_2,e_3,\ldots,e_{n-1},e_n$ can be colored in $\lambda(\lambda-1)(\lambda-2)(\lambda-3)\ldots(\lambda-(n-1))$ which can be expressed in $\lambda!$ ways. For the sequence of edges $e_1',e_2',e_3',\ldots,e_{n-1}',e_n'$, e_1' can be colored in $\lambda-2$ ways since it is adjacent to e_1 and e_2. Also, the edges e_i' can be colored in $\lambda-3$ since it is adjacent to e_i,e_{i+1} and e_{i-1}', so, the sequence of edges $e_2',e_3',\ldots,e_{n-1}',e_n'$ can be colored in $(\lambda-3)^{n-1}$ ways. The sequence of edges $u_1,u_2,u_3,\ldots,u_{n-1},u_n$ can also be colored in $(\lambda-3)^n$. Finally, by multiplication principle $f(H_n,\lambda)=\lambda!(\lambda-2)(\lambda-3)^{n-1}(\lambda-3)^n$ or $f(H_n,\lambda)=\lambda!(\lambda-2)(\lambda-3)^{2n-1}$. □

4. Conclusion and Recommendations

Graphs considered on this study are finite simple graphs. Through systematic and careful study of the properties of common classes of graphs, the following conclusions were drawn by the researcher: Chromatic index of fan graph, star graph, bistar graph, helm graph, and friendship graph were obtained and investigated. The study found out that almost all graphs are of class 1. Also, the chromatic index polynomials for the above mentioned classes of graphs were also indentified. This determines the number of ways to color the edges of a graph by using at most $\lambda-$ colors. In light of the findings of the study, the following recommendations are hereby proposed: The researcher encourages mathematics students and researchers to investigate the properties and chromatic index of other classes of graphs which has not been investigated both by foreign and local authors. Chromatic index of some classes of graphs obtained from Cartesian product and composition of two graphs are also open for future study. It is also recommended to investigate the total chromatic index, $\chi''(G)$ of common classes of graphs. Total chromatic index of a graph denotes the minimum number of colors used to color the vertices and edges of a graph such that no two adjacent vertices and edges are assigned with similar colors.

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