Characterization And Structure Of A Power Set Graph

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Abstract: This study introduced a simple graph called the Power Set Graph. This graph is taken from the lattice diagram of a power set. Let A be a finite nonempty set with cardinality n. The Power Set Graph, denoted by P, is the graph \( (\mathcal{P}(A), E(P)) \) where \( \mathcal{P}(A) \) is the power set of A and adjacency is defined by \( [X, Y] \in E(P) \) for all \( X, Y \in \mathcal{P}(A) \) if and only if \( X \subset Y \) or \( Y \subset X \). The study determined the characterization and structure of this graph including its size, order, independence number and dominance number. Results of the study showed that the power set graph is a simple graph with degree \( 2^n - 1 + q \), dominance number, \( \beta(P) = 1 \) and independence number, \( \alpha(P) = \left\lfloor \frac{n}{2} \right\rfloor \) if n is odd and \( \alpha(P) = \left\lfloor \frac{n}{2} \right\rfloor + 1 \) if n is even. Also, this paper showed the explicit formula to determine the size of the power set graph.

Keywords: Power Set Graph, Dominance Number, Independence Number.

1. Introduction
Graph theory is a branch of mathematics that focuses in the study of mathematical structured graphs used to model pairwise relations between objects. It originated as a recreational math problem, but it has grown into significant area of mathematical research with applications in various fields. Its growth has contributed in solving problems in chemical isomers, operations research, social sciences, computer science, scheduling techniques, transportation and electrical networks. It has been used to establish relationship between objects in the abstract manner and by using graph structure. For instance, graph theory was utilized to model the human protein – protein interaction networks (PPI) [1] and the constitutional isomers [2]. Also, in [3], [4] and [5], graph theory was used for transportation model and electrical networks. Aside from these applications, graph theory also provides link to other branch of mathematics such as set theory. In relation to the application of graph, this paper used graph modeling to introduce a new graph called the Power Set Graph. This graph presents the relationships among the subsets and power sets of a given set. The study aimed to; define the power set graph; determine the size of the power set graph by generalizing an explicit formula; and determine its independence number and dominance number. This study is expository in nature and pure mathematical researches were conducted. This provided comprehensive description of the concepts in mathematics, particularly on the context of graph theory and set theory. Formal definitions and theorem with formal proof of validity were utilized as the study progresses. It also employed a careful analysis of concepts and detailed discussions on power set graph were included.

2. Preliminaries
Definition 2.1[6] A graph \( G \) is an ordered pair \( (V(G), E(G)) \), where \( V(G) \) is a nonempty set of elements is called vertices and \( E(G) \) is a set of an ordered pairs of vertices called edges. A graph, \( G \) is characterized by its order (number of vertices) and size (number of edges).

Definition 2.2.[6] Let \( G \) be a graph and let \( v \in V(G) \). The degree of \( v \), denoted by \( \deg(v) \) is the number of edges incident with \( v \).

Theorem 2.1. [6] Let \( G \) be a graph with size \( m \) and order \( n \). Then \( \sum_{i=1}^{n} \deg(v_i) = 2m \).

Definition 2.3.[7] Let \( G \) be a graph and \( S \subseteq V(G) \). The set \( S \) is independent set if \( \{x, y\} \notin E(G) \) for all \( x, y \in S \).

Definition 2.4.[7] The independence number of a graph, \( G \), denoted by \( \alpha(G) \), is defined by \( \alpha(G) = \max \{\alpha\} \) where \( \alpha \subseteq V(G) \) is an independent set.

Definition 2.5.[7] Let \( G \) be a graph and \( S \subseteq V(G) \). The set \( S \) is dominating set if for all \( x \notin S \), there exists \( y \in S \) such that \( \{x, y\} \notin E(G) \).

Definition 2.6.[7] The dominance number of a graph, \( G \), denoted by \( \beta(G) \), is defined by \( \beta(G) = \min \{\beta\} \) where \( \beta \subseteq V(G) \) is a dominating set.

Definition 2.7.[8] Let \( T \) and \( S \) be sets. \( T \) is a subset of \( S \), denoted by \( T \subseteq S \), if every element of \( T \) is an element of \( S \).
Definition 2.8.[8] The Power Set of any set S, denoted by \( P(S) \), is the set of all subsets of S, including the empty set and itself.

3. Main Results

This section presents the formal definition of a power set graph, its order, size, independence number and dominance number.

Definition 3.1. Let A be a finite non-empty set with cardinality n. The Power Set Graph, denoted by P, is the graph with \( (v(P), E(P)) \) where \( v(P) \) is the power set of A and adjacency is defined by \( \{X, Y\} \in E(P) \) for all \( X, Y \in P(A) \) if and only if \( X \subset Y \) or \( Y \subset X \).

Example 3.1. Let A be a set where \( A = \{a, b\} \). The vertex set of P is \( V(P) = \{a, b, \{a\}, \{b\}, \{a, b\}, \emptyset\} \) and the edge set is \( E(P) = \{\{a, b\}, \{a\}, \{b\}, \{a\}, \{b\}, \{a, b\}, \emptyset\} \) where \( a, b \in A \). Below is the pictorial representation of P defined in example 3.1.

![Figure 3.1. Power set graph of a set with two elements.](image)

Another example of a power set graph of a set with 3 elements is given below.

Example 3.2. Let \( X = \{a, b, c\} \). The vertex set of P is \( V(P) = \{a, b, c, \{a, b\}, \{a, c\}, \{b, c\}, \{a\}, \{b\}, \{c\}, \emptyset\} \). Let \( 1 = \{a, b, c\} \); \( 2 = \{a, b\} \); \( 3 = \{a, c\} \); \( 4 = \{b, c\} \); \( 5 = \{a\} \); \( 6 = \{b\} \); \( 7 = \{c\} \) and \( 8 = \{\emptyset\} \). The pictorial representation of P is shown in figure 3.2.

![Figure 3.2. Power set graph of a set with 3 elements.](image)

Remark 3.1. The cardinality of \( v(P) \), denoted by \( |v(P)| \), is given by \( 2^n \).

Theorem 3.1. The power set graph is a simple graph.

Proof: Let A be a set with n elements and P be a power set graph where \( v(P) = P(A) \). By definition 3.1, it follows that there are no loops of the form \([A, A] \) for all \( A \in P(A) \). Also, there are no multiple edges since A is incident with \([A, B]\) once or B is incident with \([A, B]\). Since P has no loop and multiple edges, it follows that P is a simple graph.

Definition 3.2. Let A be a set with n elements. Then \( A_k \) is a collection of all subsets of A with k elements.

Remark 3.2. Let \( n, k \) be nonnegative integers such that \( 0 \leq k \leq n \). Then the cardinality of \( A_k \) is given by \( |A_k| = \binom{n}{k} \).

Lemma 3.1. For any set \( A \) with n elements, let P be a power set graph and \( B \in v(P) \). Also, let \( q_i \) be the number of edges incident with \( B \) in the form \([B, C]\) such that \( C \in v(P) \). Then \( q_i = \sum_{i=p+q-1}^{i=p} \binom{n}{i} \) such that \( 0 \leq i < r \) for some \( i, p, q \in \mathbb{Z}^+ \) where \( r = \frac{\lfloor \frac{n+1}{2} \rfloor}{2} \) if \( n \) is odd and \( r = \frac{\lfloor \frac{n}{2} \rfloor}{2} \) if \( n \) is even.

Proof: Let \( B, C \in v(P) \) be sets such that \( |C| > |\emptyset| \). Also, let \( q_i \) be the number of edges with \( B \) in the form \([B, C]\) such that \( B \subset C \). Then we have

- \( q_0 = 0 \)
- \( q_1 = \binom{1}{0} = 1 \)
- \( q_2 = \binom{2}{0} + \binom{2}{1} = 1 + 2 = 3 \)

\[
q_i = \sum_{r=1}^{n} r^{i-1} \binom{n}{i} \sum_{p=0}^{i-1} \binom{n}{p}
\]

Observe that the pattern of \( q_i \) is similar to the pattern of coefficients of the Pascal’s Triangle. Hence, \( q_i = \sum_{p+q-1}^{i=p} \binom{n}{i} \) for some \( i, p, q \in \mathbb{Z}^+ \).

Illustration 3.1. Let \( A = \{a, b, c, d, e, f, g, h, i, j\} \). Since \( n = 10 \), we have...
Theorem 3.2. Let P be a power set graph and $B \in V(P)$ such that $|B| = k$. Then $\deg(B) = 2^k - 1 + q_i$. Proof: To find the degree of B, we need to determine the number of edges incident with vertex B. From the definition of a power set graph, adjacency is defined by $[B, C] \in E(P)$ for some $C \in V(P)$ if and only if $B \subseteq C$ or $C \subseteq B$. First, determine the number of edges incident with B in the form $[B, C]$ such that $C \subseteq B$. Suppose $C \subseteq B$ for some $C \in V(P)$. It follows that $C \in P(B)$ forming an edge. The number of edges incident with B is $|P(B)|$. However, the set B is not included, so we can have $|P(B)| = 1$. Then, $|P(B)| - 1 = 2^k - 1$, since B consists of k elements. Next, from Lemma 3.1, there are $q_i$ number of edges incident with B in the form $[B, C]$ such that $B \subseteq C$. Hence, the total number of edges incident with B is $2^k - 1 + q_i$. Therefore, $\deg(B) = 2^k - 1 + q_i$.

Illustration 3.2. Let $A = \{a, b, c, d, e, f, g, h, i, j\}$ and $B = \{a, b, c, d, e, f, g, h\}$. Then $
\deg(B) = 2^5 - 1 + q_i$
$= 2^5 - 1 + (1 + 3 + 3)$
$= 262$

Theorem 3.3. Let A be a set with n – elements and P be a power set graph. Then
$n \sum_{v \in V(P)} \deg(v) = 2 \sum_{i=0}^{n} \binom{n}{i} \left(2^{n-i} - 1 + q_i\right)$ if n is odd and
$n \sum_{i=0}^{r} \binom{n}{i} \left(2^{n-i} - 1 + q_i\right) + \binom{n}{r} \left(2^r - 1 + q_r\right)$ if n is even
where $r = \left\lfloor \frac{n}{2} \right\rfloor$ if n is odd $n \geq 3$
and $r = \left\lceil \frac{n}{2} \right\rceil$ if n is even $n \geq 2$

Illustration 3.3. Let $A = \{a, b, c, d, e\}$. Since $n = 5$, then we have
$n \sum_{v \in V(P)} \deg(v) = 2\left[\binom{5}{0} \left(2^{5-0} - 1 + 0\right) + \binom{5}{1} \left(2^{5-1} - 1 + 1\right) + \binom{5}{2} \left(2^{5-2} - 1 + 2\right)\right]$
$= 2[(1)(31) + (5)(16) + (10)(10)]$
$= 422$

Theorem 3.4. Let A be a set with n – elements and P be a power set graph. Then the size of P, denoted by m is given by
$m = \sum_{i=0}^{n-1} \binom{n}{i} \left(2^{n-i} - 1 + q_i\right)$ if n is odd, $n \geq 3$ and
$m = \left[\sum_{i=0}^{n-1} \binom{n}{i} \left(2^{n-i} - 1 + q_i\right) + \binom{n}{r} \left(2^r - 1 + q_r\right)\right]$ if n is even, $n \geq 2$.

Illustration 3.4. Consider a power set graph in illustration 3.3. Since $n = 5$ and the $\sum_{v \in V(P)} \deg(v) = 422$, then by applying theorem 3.4, the size of P is $m = 211$.

Theorem 3.5. Let P be a power set graph with cardinality n. Then the independence number of P is $\beta(P) = 1$.

Proof: Suppose S is dominating set and $S \subseteq V(P)$ then we have to show that $\beta(P) = 1$. In a given set A with n number of elements, $V(P) = P(A)$ by the definition of a power set graph. Since $A \in V(P)$, we can let $A \in S$. Let A be the only element of S, then for all $x \in V(P)$ except for set A, we have $[x, A] \in E(P)$. It follows that $|S| = 1$, the possible minimum cardinality of S. Therefore, $\beta(P) = 1$

Theorem 3.6. Let P be a power set graph with cardinality n. Then the independence number of P is given by
$\alpha = \left\lceil \frac{n}{2} \right\rceil$ if n is odd
$\alpha = \left\lfloor \frac{n}{2} \right\rfloor$ if n is even

Proof: In a given set A, with cardinality n, let P be a power set graph and $A_i$ be the collections of all subsets of A with k – elements. Then $A_i$ is an independent set since for all $[x, y] \in A_i$, $[x, y] \notin E(P)$. Recall that $\alpha(P) = \max |S| : S$ is an independent set. Now, we need to determine the maximum cardinality of $A_i$. Observe that $A_0 \cup A_1 \cup A_2 \cup \ldots \cup A_n \cup A_{n+1} = P(A)$. By using Remark 3.2
$|A_i| = \binom{n}{k}$. From the properties of combinations,
$n \choose 0 = n \choose n = n \choose n-1 = n \choose 2 = n \choose n-2 \ldots n \choose r$ where $\binom{n}{r}$ is the middle. Observe that $|A_i|$ has the greatest number of elements. If n is even, then $|A_i| = \frac{n}{2}$. It implies that
\[ \alpha (P) = \left| A_\alpha \right| = \binom{n}{r} \cdot \binom{n}{\frac{n}{2}}. \] If \( n \) is odd, observe that \( \binom{n}{r} = \binom{n}{n-r} \).

Since \( \left| A_\alpha \right| = \left| A_{\alpha^-} \right| \), we choose either \( A_\alpha \) or \( A_{\alpha^-} \), as the middle term. Thus, \( \alpha (P) = \left| A_\alpha \right| = \binom{n}{\frac{n}{2}} \).

4. Conclusion and Recommendations

This paper introduced a power graph set graph taken from the power set of a given set. It is formally defined and the characterization as to its order, size, degree, dominance number and independence number are carefully investigated. Result of the study showed that a power set graph, \( P \), has a size \( m = \sum_{i=0}^{n-1} \binom{n}{n-i} (2^{n-i} - 1 + q_i) \) if \( n \) is odd, \( n \geq 3 \) and \( m = \sum_{i=0}^{n-1} \binom{n}{n-i} (2^{n-i} - 1 + q_i) + \frac{1}{2} \binom{n}{r} (2^{r} - 1 + q_r) \) if \( n \) is even \( n \geq 2 \). Also, the dominance number of this graph is 1 and its independence number is \( \alpha = \binom{n}{\frac{n}{2}} \) if \( n \) is odd and \( \alpha = \binom{n}{\frac{n}{2}} \) if \( n \) is even. In light of the findings of the study, a program algorithm to determine the size of a power set graph is open for future research. Also, the chromatic number, chromatic index and adjacency and incidence matrices of a power set graph are still open for future studies.

References


[5]. H. Robinson, “Graph Theory Techniques in Model-Based Testing”. International Conference on Testing Computer Software. 1999


[8]. L. Leithold, “College Algebra and Trigonometry” Pearson Education Asia Pte Ltd., Singapore. 2002

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