Optimality Boundary Conditions For Stochastic Elliptic System Involving Higher Order Operator

A. M. Abdallah

Basic Science Department, Higher Technological Institute,
Tenth of Ramadan, Egypt, PH-00201002621367
ahmedabdallah6236@yahoo.com

Abstract: In this present work, we deal with the stochastic elliptic systems where the model of the stochastic elliptic system is higher order operator. We prove the existence and uniqueness of the state variable process of these systems. Then, we give the necessary and sufficient for the optimality of the system.

Keywords: Optimal Stochastic Control, Optimality conditions, Stochastic analysis

1. Introduction

Here, we consider the following stochastic elliptic system involving higher order operator.

$$\begin{cases} (-\Delta)^m u(x) = W(x) \\ u(x) = \dfrac{\partial u}{\partial n} = \ldots = \dfrac{\partial^{m-1} u}{\partial n^{m-1}} = 0 \end{cases} \tag{1.1}$$

First Eq. is defined on G, Dirichlet condition is defined on ∂G, where G is a bounded, continuous domain in $\mathbb{R}^n$ with boundary ∂G, $u(x) = u(x,w) \in H_0^m(\Omega, F, P, G)$ is a state process and W(x) is a white noise.
Eq. (1.1) represents the state process equation and

$$-\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^{\,2}}$$ is the Laplacian operator.

We also prove the existence and uniqueness of the optimal stochastic control of distributed and boundary types, and we discuss the necessary and sufficient conditions of the optimality. In the following subsection, existence and uniqueness for solution of the state process equations are discussed.

1.1 Existence and Uniqueness for the state process of the system

In this subsection, we study the existence and uniqueness of solution for Eq.(1.1).

Since

$$H_0^m(\Omega, F, P, G) \subseteq L^2(\Omega, F, P, G) \subseteq H^{-m}(\Omega, F, P, G) \tag{1.2}$$

The model of the stochastic system (1.1) is given by:

$$A = (-\Delta)^m, \; Au(x) = W(x)$$

The elliptic operator A in the state equation (1.1) is a bounded second order self adjoint stochastic elliptic partial differential operator. For this operator we define the bilinear form:

$b(u,.)$ on $H_0^m(\Omega, \mathcal{F}, P; G)$, by:

$$b(u,\Phi) = (Au, \Phi)_{L^2(\Omega,F,P,G)}, \; u, \Phi \in H_0^m(\Omega, F, P, G)$$

where A maps $H_0^m(\Omega, \mathcal{F}, P; G)$ onto $H^{-m}(\Omega, \mathcal{F}, P; G)$, $H^{-m}(\Omega, \mathcal{F}, P; G)$ is the conjugate space of $H_0^m(\Omega, \mathcal{F}, P; G)$. Then,

$$b(u,\Phi) = \int_{\Omega}\int_{G} (-\Delta)^m u(x)\Phi(x)\,dx\,dp \tag{1.3}$$

and the linear form

$$L(\Phi) = \int_{\Omega}\int_{G} W(x)\Phi(x)\,dx\,dp \tag{1.4}$$

where W(x) is the white noise. Hence, the abstract Variational problem associated with Eq.(1.1) can be written as

$$\begin{cases} \text{Find } u(x) \epsilon H_0^m(\Omega, \mathcal{F}, P; G) \\ \quad \text{such that} \\ \quad b(u,v) = L(v) \end{cases}$$

By the Lax-Milgram lemma, we prove the following theorem.

Theorem 1.
The bilinear form (1.3) satisfies the stochastic coerciveness condition, and then there exists a unique solution $u(x)\epsilon H_0^m(\Omega, \mathcal{F}, P; G)$ of the system (1.1), conversely, if there exists a unique solution $u(x)\epsilon H_0^m(\Omega, \mathcal{F}, P; G)$ such that $b(u,\Phi) = L(\Phi)$, then we get the system (1.1).

Proof.
The proof of existence and uniqueness uses the Lax-Milgram lemma. It is necessary to show that the bilinear form (1.3) is continuous and coercive. We divide the proof into several steps.

(a): **Continuity**: Applying Green's formula, we find out

$$\int_G (-\Delta)^m u(x)\Phi(x)\,dx = \int_G \nabla^{\frac{m}{2}} u \nabla^{\frac{m}{2}} \Phi\,dx + \int_{\partial G} F(u,\Phi)\,d\partial G$$

Obviously, we can rewrite Eq.(1.3) by stochastic Green's formaula

$$b(u,\Phi) = E(\int_G \nabla^{\frac{m}{2}} u(x) \nabla^{\frac{m}{2}} \Phi(x)\,dx)$$

hence $F(u,\Phi) = (-\Delta)^{m-1} u \dfrac{\partial \Phi}{\partial n} = 0$

on ∂G by virtue Dirchlet condition. Then, by Cauchy Schwartz inequality

$$\left|b(u,\Phi)\right| \le E\left(\int_G \nabla^{\frac{m}{2}} u(x)dx\right)^{\frac{1}{2}} E\left(\int_G \nabla^{\frac{m}{2}} \Phi(x)dx\right)^{\frac{1}{2}},$$

since

$$\left\|u\right\|_{H_0^m(\Omega,F,P,G)} = E\left(\int_G \left|\nabla^m u(x)\right|^2 dx\right)^{\frac{1}{2}},$$

then

$$\left|b(u,\Phi)\right| \le \left\|u\right\|_{H_0^m(\Omega,F,P,G)} \left\|\Phi\right\|_{H_0^m(\Omega,F,P,G)}, \tag{1.5}$$

Thus, the linear form L(.) is continuous on

$H_0^m(\Omega,F,P,G)$, by using Cauchy Schwartz inequality, we find out

$$L(\Phi) = \int_\Omega \int_G W(x)\Phi(x)dxdp$$

$$\le \int_\Omega \left(\int_G (W(x))^2 dx\right)^{\frac{1}{2}} dp \int_\Omega \left(\int_G (\Phi(x))^2 dx\right)^{\frac{1}{2}} dp$$

$$= \left\|W\right\|_{L^2} \left\|\Phi\right\|_{L^2}$$

from Eq.(1.2), we have

$$\left\|u\right\|_{L^2(\Omega,F,P,G)} \le \left\|u\right\|_{H_0^m(\Omega,F,P,G)},$$

Then, we get

$$L(\Phi) \le \left\|W\right\|_{H_0^m} \left\|\Phi\right\|_{H_0^m} \tag{1.6}$$

(b): **Coerciveness**: From definition of the norm on

$$H_0^m(\Omega, F, P, G)$$

$$\left\|u\right\|^2_{H_0^m} = E\left(\int_G \left|\nabla^m W\right|^2 dx\right)$$

$$= E\left(\int_G \nabla^m W \nabla^m W\, dx\right)$$

$$\le E\left(\int_G \nabla^m W \nabla^m W\, dx\right) + E\left(\int_{\partial G} F(u,\Phi)d\partial G\right)$$

By Green's formula, we obtain

$$\left\|u\right\|^2_{H_0^m} \le E\left(\int_G (-\Delta)^m u.u dx\right) = b(u,u)$$

Therefore,

$$b(u,u) \ge c\left\|u\right\|^2_{H_0^m} \quad \textbf{(Stochastic coerciveness)} \tag{1.7}$$

It is easy to construct the following Sobolev spaces $\left[H_0^m(\Omega,F,P,G)\right]^2$ by the 2-times Cartesian product as follows:

$$\left[H_0^m(\Omega,F,P,G)\right]^2$$
$$= \left[H_0^m(\Omega,F,P,G)\right] \times \left[H_0^m(\Omega,F,P,G)\right].$$

Since the bilinear form b(u,.) is continuous and stochastic coercive on $\left[H_0^m(\Omega,F,P,G)\right]^2$, and the linear form is also continuous on $\left[H_0^m(\Omega,F,P,G)\right]^2$, by Lax Milgram lemma there exist a unique solution $u \in \left[H_0^m(\Omega,F,P,G)\right]^2$, such that

$$b(u,\Phi) = L(\Phi)\forall \Phi \in [H_0^m(\Omega,F,P;G)]^2. \tag{1.8}$$

Conversely, when $b(u,\Phi) = L(\Phi)\forall \Phi \in [H_0^m(\Omega,F,P;G)]^2.$ and $u \in [H_0^m(\Omega,F,P;G)]^2,$

integrating Eqn. (1.1) on G and taking expectation, we find

$$\int_\Omega \int_G \nabla^m u \nabla^m \Phi\, dxdp = \int_\Omega \int_G W(x)\Phi(x)dxdp$$

By applying stochastic Green's formula:

$$\int_\Omega \int_G (-\Delta)^m u\Phi\, dxdp + \int_\Omega \int_{\partial G} F(u,\Phi)d\partial G dp = \int_\Omega \int_G W(x)\Phi(x)dxdp,$$

on ∂G and $F(u(x), \Phi(x)) = 0$ by virtue of Dirichlet condition $u(x) = 0$. By comparison of two sides, we deduce the system (1.1), which completes the proof. Under the above consideration, using the theorems 1.1, 1.2 of [4], we can formulate the following Dirichlet problem, which define the state process of our control problem. Now, we formulate the control problem with adding the control in the region G and we determine the cost functional.

1.2 Formulation of the boundary control problem with boundary observation

We add the control on the boundary to formulate the initial boundary value Dirichlet problem for stochastic elliptic systems and we investigate necessary conditions for an optimal boundary control problem. The space $L^2(\Omega, F, P; \partial G)$ is the space of controls. For a control $y \in L^2(\Omega, F, P; \partial G)$, the state process u of the system is given by the solution of the following system:

$$\begin{cases} (-\Delta)^m u(y) = W(y) \\ u(y) = \dfrac{\partial u}{\partial n} = \dots = \dfrac{\partial^{m-1} u}{\partial n^{m-1}} = y \end{cases} \tag{1.9}$$

The observation equation is given by $\chi(y) \equiv u(y)$, and the cost functional is given by:

$$C(y) = E\left[\int_{\partial G} (u(y) - \chi_d)dx\right] + \int_\Omega \int_{\partial G} Mzydxdp \tag{1.10}$$

where $\chi_d \in L^2(\Omega, \mathcal{F}, P; G)$.
Then, the control problem is defined by $y \in Y_{ad}$ such

that:

$$C(y) \le C(z) \forall z \in Y_{ad};$$

where Y_{ad} is a closed convex subset from $L^2(\Omega, \mathcal{F}, P; \partial G)$. The cost functional (1.10) can be rewritten as:

$$C(y) = \int_{\Omega}\int_{\partial G} (u(y) + u(0) - u(0) - \chi_d)^2 dx dp + \int_{\Omega}\int_{\partial G} M y z dx dp$$

where

$$\pi(y,z) = \int_{\Omega}\int_{\partial G} ((u(y) - u(0))^2 + (u(z) - u(0))^2) dx dp$$

$$+ \int_{\Omega}\int_{\partial G} M y z dx dp \qquad (1.11)$$

$$L(z) = \int_{\Omega}\int_{\partial G} ((\chi_d - u(0))^2 + (u(z) - u(0))^2) dx dp \qquad (1.12)$$

Theorem 2.
If the state $u(y)$ is given by Eq.(1.1) and if the cost functional is given by Eq.(1.10), then there exists a unique optimal control $y \in Y_{ad}$ such that

$C(y) \le C(z) \forall z \in Y_{ad};$

Moreover, it is characterized by the following equation

$$\begin{cases} (-\Delta)^m h(y) = u(y) - \chi_d \\ h(y) = 0 \end{cases}$$

The inequality $\int_{\Omega}\int_{\partial G} (\frac{\partial h}{\partial v_A} + My)(z - y) dx dp \ge 0$

represents the necessary and sufficient condition for optimality equation.

Proposition 1.

If the constraints are absent, i.e., when $Y = Y_{ad}$ **then the equality** $h + My = 0, z \ne y$ **the differential problem of finding the vector-function satisfies the following relations, the equation**

$$\begin{cases} Au = W \\ u + \dfrac{h}{M} = 0 \end{cases} \qquad \textbf{(1.13)}$$

represents the state process equation for the system (1.1) without constraints.

Example 1.

For m = 1 , it is proved in [5] that the state of the system is given

$$\begin{cases} (-\Delta) u(y) = W(y) \\ u(y) = 0 \end{cases}$$

$$b(u, \Phi) = \int_{\Omega}\int_{G} -\Delta u(x) \Phi(x) dx dp$$

Example 2.

For m = 2 , it is proved that the state of the system is given

$$\begin{cases} \Delta^2 u(y) = W(y) \\ u(y) = 0 \end{cases}$$

$$b(u, \Phi) = \int_{\Omega}\int_{G} \Delta^2 u(x) \Phi(x) dx dp$$

2. Acknowledgments

The author would like to express his gratitude to professor Dr **Ahmed Said Okb El Bab**, Department of Mathematics, Al Azhar Univeristy, Cairo, Egypt and Professor Dr **Claudio Roberto Ávila,** Departamento Acadêmico De Mecânica, Curitiba, Brazil for suggestion the problem and critically reading the manuscript.

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Author Profile

Author received the B.S. and M.S. degrees in Functional Analysis (Optimal Control) from Zagazig Univeristy, Al Azhar Univeristy in 2007 and 2017, respectively. During 2009-now, he stayed in Basic Science Department, Higher Technological Institute, Tenth of Ramadan City, Egypt.