A Non-Linear Partial Differential Accretive Operator In A Closed, Bounded And Continuous Domain

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ABSTRACT: A non-linear parabolic partial differential equation is investigated in a closed, bounded and continuous domain by converting such an equation into an abstract Cauchy problem. Using some results established in Egwurute and Garba (2003), this operator is shown to be m-accretive thus establishing that this partial differential equation has a solution by the fundamental results of Browder (1967) on the theory of accretive operators.

Keywords: Accretive, operator, non-linear partial differential equation, Cauchy problem

1. INTRODUCTION
Partial differential equations (PDE) help in revealing hidden connections that lead to rational explanations of some physical phenomena, which was the major attraction for this project. It is possible to propose some special solutions to partial differential equations explicitly in terms of elementary functions. One method of finding such explicit solution to partial differential equation is by reducing it to first order ordinary differential equation, which then can be solved using the standard Picard-Lindelof theorem. The issue of existence and uniqueness of solutions of ordinary differential equations have a very satisfactory answer in the Picard-Lindelof theorem, which is not the case for partial differential equations. [1] Demonstrated that the existence and uniqueness of solutions to differential equations satisfying Lipschitz conditions satisfied a more general condition for existence and uniqueness than the requirement of the continuity of $\frac{\partial f}{\partial x}$.

Furthermore they postulated the use of accretive operator for differential equation defined on a non-bounded domain and established the necessary conditions for existence and uniqueness of solutions. Another method known as the semi-group approach, has significantly clarified and unified the study of many classes of partial and functional differential equations and have solved problems that were left open by the previous methods. The importance of the notion of “accretive mapping” subsists in the fact that it allows for the treatment of many partial differential equations and functional differential equations in mathematical physics, for example such as the heat and wave equations. There are two main problems associated with the class of accretive operators: the time-dependent problem and the autonomous (time-independent) problem. The time-dependent problem is usually concerned with the solvability of first and second order evolution equations, while the autonomous problem deals with equations of elliptic type. The elliptic type (elliptic partial differential equations) problems often involve sums of time independent operators, some of which may be accretive or M-accretive [6]. The theory of M-accretive operators parallels the theory of maximal monotone operators in appropriate spaces. There are a few basic properties of maximal monotone operators that do not yet have a counterpart in the class of the M-accretive operators. [4], showed that if $X$ is a reflexive Banach space, $X^*$ its dual space, the pairing between $u$ in $X$ and $w$ in $X^*$ is written as $(w, u)$. If $T$ is a mapping from $X$ to $2^{X^*}$, (where $2^{X^*}$ is the family of all subsets of $X^*$), its domain $D(T) = \{u|T(u) \neq \emptyset\}$, $R(T) = U_{u \in X} T(u)$. $T$ is said to be monotone if for $w \in T(u), z \in T(x)$, then $(w - z, u - x) \geq 0$. According to [3], reflexive Banach spaces have important properties in geometry and in operator-theoretic settings; in geometric setting, given any closed convex subset $A$ of a reflexive Banach space $X$ and any point $x \in X$, the norm defined on $X|A$ attains its minimum on $A$ if $\|X\| = \inf_{x \in A} |x - d|$. In fact, this is a special case of a more general theorem which states that continuous convex functions on a reflexive spaces attain their extrema over a convex subspace. The operator-theoretic setting describes how the process of taking adjoints of bounded operators defined on a reflexive Banach spaces $X$ shows that every bounded linear operator is its own double adjoint if and only if $X$ is a reflexive space. This work considered “a nonlinear partial differential accretive operator defined in the closed, bounded and continuous domain”, which was then show that it has solution. [5] investigated a class of quasi-linear hyperbolic first order partial differential equation (PDE) in a Banach space, by converting it to Cauchy-like problem and thereafter established that it is m-accretive. [1] used the same technique on a class of quasi-linear parabolic
partial differential equations before arriving at the fact that it is also m-accretive. A class of nonlinear partial differential equations (PDE) defined in an appropriate Banach space is investigated for accretiveness condition and existence of solution. This article investigates the existence of solution of a nonlinear parabolic partial differential equation (PDE) with the objectives:
(a) to find an appropriate Banach space for this problem,
(b) to investigate the continuity and accretive property of the equivalent Cauchy problem,
(c) to establish the existence of solution of the partial differential equation (PDE).

2. Methodology
Let \( X \) be a real Banach space with norm \( \| \cdot \| \) and dual \( X^* \). An operator \( A \) with domain \( D(A) \) and range \( R(A) \), in \( X \) is said to be accretive, if for all \( x_1, x_2 \in D(A) \) and \( r > 0 \), there holds the inequality [2].

\[
\| x_1 - x_2 \| \leq \| x_1 - x_2 + (A x_1 - A x_2) \| \quad (1)
\]

An accretive operator \( A \) is said to be m-accretive if \( R(I + ra) = X \) for all \( r > 0 \), where \( I \) is the identity operator on \( X \). In terms of the concept of contractions, an operator \( A \) is said to be accretive if \( R(I + rA)^{-1} \) is a contraction for \( r \geq 0 \) that is if

\[
\| x_1 + rAX_1 - (x_2 + rAX_2) \| \geq \| x_1 - x_2 \| \quad (2)
\]

If \( X \) is a Hilbert space, the accretive condition (2) reduces to

\[
\Re \langle Ax_1 + Ax_2, x_1 + x_2 \rangle \geq 0 \quad \text{for all} \quad x_1, x_2 \in X \quad (3)
\]

Theorem [5]
Let \( X \) be a real Banach space and let \( A : D(A) \subset X \to X \) be an m-accretive operator such that

\[
u = (A + n^{-1}I)x_1, v = (A + n^{-1}I)x_2
\]

where \( u, v \in X; x_1, x_2 \in D(A); n \in R^+ \)

Then operator \( (A + n^{-1}I) : X \to D(A) \) is continuous and bounded. Let reproduce the proof in [5], for completion purposes

Proof:
Given \( u, v \in X; x_1, x_2 \in D(A); n \in R^+ \) as defined above we have

\[
\langle (A + n^{-1}I)x_1 + (A + n^{-1}I)x_2, j \rangle \leq \frac{1}{n} \| x_1 - x_2 \|^2 \quad \text{for} \quad j \in J(x_1 - x_2)
\]

This implies that

\[
\| (A + n^{-1}I)^{-1}u - (A + n^{-1}I)^{-1}v \| \leq n \| u - v \|,
\]

which shows the continuity of \( (A + n^{-1}I)^{-1} \)

Now let \( v = 0, x_0 \in D(A) \) with \( 0 = (A + n^{-1}I)x_0 \), then we have

\[
\left\| (A + n^{-1}I)^{-1}u \right\| \leq n \left\| u \right\| + \| x_0 \|
\]

This proves the boundedness of the operator \( (A + n^{-1}I)^{-1} \). This ends the proof.

3. RESULTS
Existence of Solution of Non-Linear Parabolic Equation
Considering a nonlinear parabolic partial differential equation defined on a closed, bounded and continuous domain (Space of \( C[0,1] \)).

\[
\phi(u_t) - u_{xx} = 0, \quad t > 0, \quad 0 < x < 1
\]

\[
u(0, x) = u_0(x), \quad 0 < x < 1 \quad (4)
\]

\[
u(t, 0) = u(t, 1) = 0, \quad t > 0
\]

Where \( \phi : R \to R \) is continuous, non-decreasing and \( \phi(0) = 0 \), which we transform into the initial value problem

\[
\frac{du}{dt} + Au = 0, \quad u(0) = u_0 \quad (5)
\]

defined on a Banach space \( X = C[0,1] \) assuming that \( u_{xx} \) is continuous, strictly increasing, \( u_{xx}(0) = 0 \), \( u_{xx}(R) = R \) and consequently \( A \) is m-accretive.

Preposition:
Let \( X = C([0,1]), D(A) = \{ u \in C([0,1]) : u, u', u'' \} \) are continuous and \( u(0) = u(1) = 0 \) \( \) and \( A(u) = \{ v \in C[0,1] : \phi(-v) = u'' \} \) for \( u \in D(A) \);

where \( V = u_t \), then \( A \) is m-accretive.

Proof:
We shall first show that the operator \( A \) is accretive. Since \( \phi : R \to R \) is continuous, non-decreasing with \( \phi(0) = 0 \), we define the associated operator \( A \) as a single-valued function. Suppose \( u_1, u_2 \in D(A), \) and \( v_1 \in Au_1, v_2 \in Au_2 \)

Let \( u_1 \neq u_2 \) then

\[
\| u_1 - u_2 \| \leq \max_{0 \leq x \leq 1} \| v_1(x) - v_2(x) \| = u_1(x_0) - u_2(x_0)
\]

Since \( u_1 = u_2 = 0 \) at \( x = 0 \) and \( x = 1 \Rightarrow 0 < x_0 < 1 \)

Case 1:
If \( v_1(x_0) \leq v_1(x_0) \) and \( \lambda > 0 \) then

\[
\| (A + n^{-1}I)^{-1}u - (A + n^{-1}I)^{-1}v \| \leq n \| u - v \|,
\]
\[
\max_{0 \leq t \leq 1} |u_t(x) - u_2(x) + \lambda(v_1(x) - v_2(x))| \\
\geq (u_1(x_0) - u_2(x_0) + \lambda(v_1(x_0) - v_2(x_0))
\]

**Case 2:**

If \( v_2(x_0) > v_1(x_0) \)

\[ \iff -v_1(x_0) > -v_2(x_0) \]

Therefore by the monotonicity of \( \phi \)

\[ \Rightarrow \phi(-v_2(x)) \leq \phi(-v_1(x)) \text{ on } [x_0 - \delta, x_0 + \delta^2] \text{ where } \delta > 0 \]

Consequently

\[ (u_1 - u_2)\,'(x) = \phi(-v_1(x) - \phi(-v_2(x)) \geq 0 \]

on \([x_0 - \delta, x_0 + \delta]^2\]

Hence \( u_1 - u_2 \) is convex on \([x_0 - \delta, x_0 + \delta]^2\) and therefore assumes its maximum at the interior point \( x_0 \).

Then \( u_1 - u_2 \) is constant \( x_0 - \delta \leq x \leq x_0 + \delta \). If \( x_0 \) is the least number in \([0,1]\) for which case (1) holds leads to a contradiction that \( v_2(x_0) > v_1(x_0) \iff u_1 - u_2 < 0 \).

Hence \( v_1(x_0) \geq v_2(x_0) \).

Using definition of accretive and the concepts of accretive operator in condition (1) then case 1 and 2 shows that \( A \) is accretive. Now it remains to show that it is m-accretive, that is, \( R(I + A) = C[0,1] \), where \( I \) is the identity operator, \( R \) is range, and \( r \) is assumed to equal to 1. Consider, \( h \in C[0,1], u \in D(A) \) such that

\[ h \in (I + A)u \text{ or } h - u = Au. \]

for \( u = \phi(v) \) and \( \beta(u) = \phi^{-1} \) means that the setting

\[ u(0) = u(1) = 0 \text{ and } u\,' = \phi(u - h) \]

can be solved.

Since \( A \) is accretive then if \( \beta(u) \in D(A) \) and \( \beta(u) = u\,' - h \), then

\[ \|\beta(u)\|_e \leq \|\beta(u) - \beta(u) - 0\|_e = \|h\|_e = \max_{0 \leq t \leq 1}|h| \]

Also by the mean value theorem, for some \( \xi \in [0,1] \),

\[ |u' (x)| \leq \|\int_0^1 u' (s) ds\| \leq \|\int_0^1 |u' (s)| ds\| \leq 2\|h\|_{C([0,1])} \] and

\[ |u(x)| \leq \|\int_0^x u' (s) ds\| \leq \|\int_0^1 |u' (s)| ds\| \leq 2\|h\|_{C([0,1])} = \max_{0 \leq t \leq 1}|h| \]

Since the operator \( A \) has been shown to be accretive, and since \( R(I + A) = C[0,1] \), then it is m-accretive. Hence by [2], this operator \( A \) for nonlinear parabolic problem admits a solution. Hence a solution for this nonlinear parabolic partial differential equation problem. This ends the proof.

**4. Discussion of the Results**

Here, \( X = C[0,1] \) is the space of closed bounded and continuous domain, we define the domain of the nonlinear operator as a dense subset of \( X \); under the conditions given in Preposition, our non-linear operator is not only accretive but also m-accretive. It has been established that the non-linear partial differential equation of the type in equation (4), has a Cauchy-like equation (5) where the operator which is defined in the space of bounded, continuous domain is m-accretive. It is known that if \( X \) is a Banach space (Hilbert space), then an accretive mapping of \( X \) into itself is locally bounded at every point of the interior of its domain. We had shown that this is also true in case 1 and 2 i.e the proof of preposition provided that the duality mapping of \( X \) is weakly continuous. We used this fact to show that, under some stated conditions, an accretive mapping is m-accretive if and only if it is maximally accretive i.e. the \( R(I + rA) = C[0,1] \), where \( r > 0 \).

**5. Conclusion**

The m-accretiveness of the non-linear operator in space of continuous domain problem has been established, just as it was observed in the hyperbolic case as by [5]. We have established in this work that, a nonlinear parabolic partial differential equation of the kind in equation 4 defined on a Banach space \( X = C[0,1] \) is continuous, accretive and thus admit a solution. The analysis of this non-linear parabolic partial differential equation problem for accretive properties will no doubt help mathematics educators, researchers and scientists who are interested in nonlinear systems.

**REFERENCES**


