Erdős - Faber - Lovasz Graphs

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Abstract: A famous conjecture of Erdős, Faber, and Lovasz states that if the edge set of a graph $G$ can be covered with $n$ copies of $K_n$, the complete graph on $n$ vertices, such that two of this $K_n$ share at most one vertex, then the chromatic number $\chi(G)$ of $G$ is just $n$. The best upper bound so far has been proved by W. I. Chang and E. Lawler in [2]. A graph $G$ is said to be decomposable into subgraphs $G_1, G_2, \ldots, G_k$ of $G$ if any two subgraphs $G_i$ and $G_j$ have no edges in common and the union of all subgraphs $G_i$, $1 \leq i \leq k$, is $G$. We say that graph $G$ is an Erdős-Faber-Lovasz graph (EFL) if $G$ is decomposable into $k$ complete graphs on $k$ vertices. That is, $G = \bigcup_{i=1}^{k} G_i$ where each $G_i$ is a complete graph with $k$ vertices. We call each $G_i$ the summand of $G$. If $G$ is an EFL graph then it follows from the definition of a decomposable graph that every pair $(G_i, G_j)$ of $G$ has at most one common vertex. We say that a vertex $v$ of $G$ a bridge point if there exist $G_i$ and $G_j$ such that $v \in V(G_i) \cap V(G_j)$.

Keywords: bridge point, component-bridge point transformation, decomposable graph, Erdős-Faber-Lovasz graph

Results and Discussions

Proposition 1: If $G$ is an EFL graph, then each pair of components of $G$ has at most one bridge point.

Proof:
Let $G$ be an EFL graph. Then $G = \bigcup_{i=1}^{k} G_i$ for some $k$ and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. If each $G_i$ has no bridge point, then we are done. Suppose there is a pair $(G_i, G_j)$ with more than one bridge point. Then $G_i$ and $G_j$ share a common edge. This is a contradiction of being an EFL graph.

Definition: An EFL graph in which all $k$-components has exactly $i$ bridge points where $i \geq 1$ is called a $(k \times i)$ – EFL graph. Specifically, when $i = 1$ and $i = 2$, the graphs are called EFL $k$-star and EFL $k$-ring, respectively.

The following are some examples of such graphs.

![Figure 1: EFL k – star](image1)

![Figure 2: EFL k – ring](image2)

![Figure 3: EFL k – graphs](image3)

Proposition 2: If $G$ is an EFL $k$-ring, then there exist a unique cycle in $G$ of length $k$.

Proof:
Suppose $G$ is an EFL $k$-ring. Let $G_1, G_2, G_3, \ldots, G_{k-1}$ be the decomposition of $G$. WLOG, let $(G_1, G_2), (G_2, G_3), \ldots, (G_{k-1}, G_k), (G_k, G_1)$ have distinct bridge points say $v_1, v_2, v_3, \ldots, v_{k-1}, v_k, v_1$ is a cycle of length $k$. It is easy to see that this cycle is unique.

Proposition 3:
If $G$ is an EFL $k$ – ring, then $G$ is Hamiltonian.

Proof: (Follows from Proposition 2)
It follows from the definition of a $(k \times i)$ – EFL graph that there does not exist an EFL graph such that $k \leq i$ since the maximum number of bridge points of an EFL graph is $k = 1$. Furthermore, we can treat each component of $G_i$ of an EFL $k$ – ring as a vertex and the bridge point $v$ as an edge joining $G_i$ and $G_{i+1}$ whenever $v$ is a bridge point of the pair, then we can view EFL $k$ – ring graph simply as a cycle. We will call the graph obtained by this process as component-bridge point transformation (‘com-bridge transformation”) of an EFL graph. That means, that in this transformation of an EFL graph, the number of edges corresponds to the number of bridge points.

Proposition 4:
A $(k \times i)$ – EFL graph where $i \geq 3$ and $k > i$ can be constructed from an EFL $k$ – star or EFL $k$ – ring.

Proof:
We want to show that for $k \geq 3$ and given any $i$, the number of bridge points of the graph constructed from either an EFL $k$ – star or EFL $k$ – ring is $i$. Thus the proof will make use of the following cases:

Case I. When $i$ is even.
Since and EFL $k$ – ring has exactly two bridge points, thus the construction will be based on it for us to get an even number of bridge points. The graph of a $(k \times i)$ – EFL is obtained by connecting the components $G_i$ and $G_{(i+1) \mod k}$ of a com-bridge transformation of an EFL graph where $1 < s \leq \frac{i}{2}$. The following length of cycles are obtained:

\[
\begin{align*}
    s = 2 & \rightarrow s_1 \text{ cycles of length } l_1 \\
    s = 3 & \rightarrow s_2 \text{ cycles of length } l_2 \\
    s = \frac{i}{2} & \rightarrow s_1 \text{ cycles of length } l_1^1 \\
\end{align*}
\]
where \((s_i)(l_t) = k\) for \(1 \leq t \leq \frac{i}{2} - 1\).

Since it is an EFL \(k\) - ring graph, then the total number of edges is \(2 \left(k \left(\frac{i}{2} - 1\right) + k\right) = ki\). Hence the number of bridge points is \(\frac{ki}{k} = i\). (NOTE: We are not concern with what are the values of \(s_i\) and \(l_t\) since their product is always equal to \(k\).)

**Case II. When \(i\) is odd and \(k\) is odd.**

The graph of a \((k \times i) - EFL\) is obtained by connecting the components \(G_j\) and \(G_{(j+1)mod k}\) of an EFL \(k\) - star graph where \(1 < s \leq \left\lfloor \frac{i}{2}\right\rfloor\).

\[
\begin{align*}
  s = 2 & \rightarrow s_1 \text{cycles of length } l_1 \\
  s = 3 & \rightarrow s_2 \text{cycles of length } l_2 \\
  s = \frac{i}{2} & \rightarrow s_{\frac{i}{2}} \text{cycles of length } l_{\frac{i}{2}}
\end{align*}
\]

where \((s_i)(l_t) = k\) for \(1 \leq t \leq \frac{i}{2} - 1\).

Since such a graph is an EFL \(k\) - star, hence the total number of edges is \(2 \left[k \left(\frac{i}{2} - 1\right) + k\right] = ki\). Thus, the number of bridge points is \(\frac{ki}{k} = i\).

**Case III. When \(i\) is odd and \(k\) is even.**

From a com-bridge point transformation of an EFL graph, construct such \((k \times i) - EFL\) by joining the components \(G_j\) and \(G_{(j+\frac{1}{2})mod k}\) and \(G_j\) and \(G_{(j+\frac{1}{2})mod k}\) where \(1 < s \leq \left\lfloor \frac{i}{2}\right\rfloor\).

\[
\begin{align*}
  s = 2 & \rightarrow s_1 \text{cycles of length } l_1 \\
  s = 3 & \rightarrow s_2 \text{cycles of length } l_2 \\
  \vdots \\
  s = \frac{i}{2} & \rightarrow s_{\frac{i}{2}} \text{cycles of length } l_{\frac{i}{2}}
\end{align*}
\]

where \((s_i)(l_t) = k\) for \(1 \leq t \leq \frac{i}{2} - 1\).

Since such a graph is an EFL \(k\) - star, hence the total number of edges is \(2 \left[k \left(\frac{i}{2} - 1\right) + k\right] = ki\). Thus, the number of bridge points is \(\frac{ki}{k} = i\).

**Proposition 5:**

The com-bridge transformation of a \((k \times i) - EFL\) graph, where \(k - i = 1\) for \(i \geq 3\), is a complete graph on \(k\) vertices.

**Proof:**

Let \(G\) be a \((k \times i) - EFL\) graph where \(k - i = 1\) for \(i \geq 3\), that is, \(G\) has \(k\) components with exactly \(i\) bridge points. By the hypothesis, \(k - i = 1\) implies \(i = k - 1\). Hence, \(G\) has \(k\) components with exactly \(k - i\) vertices each. Therefore, \(G\) is a complete graph on \(k\) vertices.

**Proposition 6:**

If \(G\) is a \((k \times i) - EFL\) graph, then \(G\) is \(k\) colorable.

**Proof:**

Since every component of a \((k \times i) - EFL\) graph is a complete graph on \(k\) vertices, then each \(k\) components is \(k\) colorable, that is, every \(G_j\) is \(k\)-colourable. Now, since \(i < k\), hence a greedy coloring can be performed to color the \(i\) bridge points of each component from the available \(k\) colors and the remaining vertices of each component which are not bridge point/s can be colored properly by the remaining \(k - i\) colors. Thus, it is not possible to color \(G\) by \(k + 1\) colors.

**Definition:**

A \(k - \text{fan}\) graph is an EFL graph with \(k\) components, \(k - 1\) of which all share a common bridge point, and each exactly one other bridge point.

Some examples of \(k - \text{fan}\) graphs.

![Figure 4: EFL k – fan graphs](image)

It is clear from the definition above that since \(k - 1\) components share a common bridge point, hence each of them should be connected to the \(k^{th}\) component for these \(k - 1\) components have exactly two bridge points. Thus, the \(k^{th}\) component has \(k - 1\) bridge points.

**Proposition 7:**

A \(k - \text{fan}\) graph \(G\) is \(k\) colorable.

**Proof:**

Let \(G\) be a \(k - \text{fan}\) graph. It implies that the \(k^{th}\) component has exactly \(1\) bridge points and the \(k^{th}\) component is \(k\) colorable. Hence color these \(k - 1\) bridge points from the available \(k\) colors and color the common bridge point of the \(k - 1\) components by the remaining \(1\) color of the \(k^{th}\) component. Thus, each of the two bridge points of the \(k - 1\) components are colored already. So, color the remaining vertices of the \(k - 1\) components (which are not bridge points) by the remaining \(k - 2\) available colors of each component.

**References**


[3]. Erdös, Paul, “On the combinatorial problems which I most like to see solved.”, Combinatorica 1, pp. 25 – 42, 1981


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