Erdös - Faber - Lovasz Graphs

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Abstract: A famous conjecture of Erdös, Faber, and Lovasz states that if the edge set of a graph G can be covered with n copies of K_n, the complete graph on *n* vertices, such that two of this K_n share at most one vertex, then the chromatic number χ(G) of G is just n. The best upper bound so far has been proved by W. I. Chang and E. Lawler in [2]. A graph G is said to be ***decomposable*** into subgraphs $G_1, G_2, \dots,$ G_k if any two subgraphs G_i and G_j have no edges in common and the union of all subgraphs G_i, $1 \le i \le k$, is G. We say that graph G is an Erdös-Faber-Lovasz graph (**EFL**) if G is decomposable into k complete graphs on k vertices. That is, $G = \cup_{i=1}^{k} G_i$ where each G_i is a complete graph with k vertices. We call each G_i the summand of G. If G is an EFL graph then it follows from the definition of a decomposable graph that every pair (G_i, G_j) of G has at most one common vertex. We say that a vertex v of G a ***bridge point*** if there exist G_i and G_j such that $v \in V(G_i) \cap V(G_j)$.

Keywords: bridge point, component-bridge point transformation, decomposable graph, Erdös-Faber-Lovasz graph

Results and Discussions

Proposition 1: If G is an EFL graph, then each pair of components of G has at most one bridge point.

Proof:
Let G be an EFL graph. Then $G = \cup_{i=1}^{k} G_i$ for some k and $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. If each G_i has no bridge point, then we are done. Suppose there is a pair (G_i, G_j) with more than one bridge point. Then G_i and G_j share a common edge. This is a contradiction of being an EFL graph.

Definition: An EFL graph in which all k-components has exactly i bridge points where $i \geq 1$ is called a $(k \times i) - EFL$ graph. Specifically, when $i = 1$ and $= 2$, the graphs are called **EFL k-star** and **EFL k-ring**, respectively.

The following are some examples of such graphs.

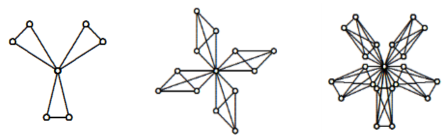

Figure 1: EFL k – star

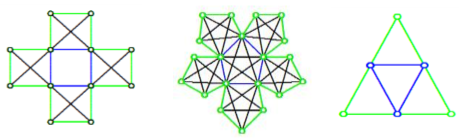

Figure 2: EFL k – ring

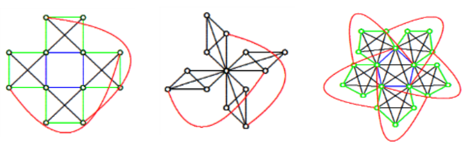

Figure 3: EFL k – graphs

Proposition 2:
If G is an EFL k-ring, then there exist a unique cycle in G of length k.

Proof:
Suppose G is an EFL k-ring. Let $G_1, G_2, G_3, \dots, G_{k-1}$be the decomposition of G. WLOG, let (G_1, G_2), (G_2, G_3), … , (G_{k-1}, G_k), (G_k, G_1) have distinct bridge points say $v_1, v_2, v_3, \dots, v_{k-1}, v_k, v_1$ is a cycle of length k. It is easy to see that this cycle is unique.

Proposition 3:
If G is an EFL k – ring, then G is Hamiltonian.

Proof:
(Follows from Proposition 2)
It follows from the definition of a (k x i) – EFL graph that there does not exist an EFL graph such that $k \leq i$ since the maximum number of bridge points of an EFL graph is $k - 1_1$. Furthermore, we can treat each component of G_j of an EFL k – ring as a vertex and the bridge point v as an edge joining G_j and G_{j+1} whenever v is a bridge point of the pair, then we can view EFL k – ring graph simply as a cycle. We will call the graph obtained by this process as component-bridge point transformation (*'com-bridge transformation"*) of an EFL graph. That means, that in this transformation of an EFL graph, the number of edges corresponds to the number of bridge points.

Proposition 4:
A $(k \times i)$ – EFL graph where $i \geq 3$ and $k > i$ can be constructed from an EFL k – star or EFL k – ring.

Proof:
We want to show that for $k \geq 3$ and given any i, the number of bridge points of the graph constructed from either an EFL k –star or EFL k – ring is i. Thus the proof will make use of the following cases:

Case I. When *i* is even.
Since and EFL k – ring has exactly two bridge points, thus the construction will be based on it for us to get an even number of bridge points. The graph of a $(k \times i) - EFL$ is obtained by connecting the components G_j and $G_{(j+s) mod\, k}$ of a com-bridge transformation of an EFL graph where $1 < s \leq \frac{i}{2}$. The following length of cycles are obtained:

$$s = 2 \rightarrow s_1 \text{ cycles of length } l_1$$
$$s = 3 \rightarrow s_2 \text{ cycles of length } l_2$$

$$s = \frac{i}{2} \rightarrow s_{\frac{i}{2}-1} \text{ cycles of length } l_{\frac{i}{2}-1}$$

where $(s_t)(l_t) = k \; for \; 1 \le t \le \frac{i}{2} - 1$.
Since it is an EFL k – ring graph, then the total number of edges is $2\left[k\left(\frac{i}{2} - 1\right) + k\right] = ki$. Hence the number of bridge points is $\frac{ki}{k} = i$. *(NOTE: We are not concern with what are the values of* s_t *and* l_t *since their product is always equal to* k*.)*

Case II. When *i* is odd and *k* is odd.
The graph of a $(k \times i) - EFL$ is obtained by connecting the components G_j and $G_{(j+s) mod\, k}$ of an EFL k – star graph where $1 < s \le \left\lceil \frac{i}{2} \right\rceil$.

$$s = 2 \;\rightarrow\; s_1 \; cycles \; of \; length \; l_1$$
$$s = 3 \;\rightarrow\; s_2 \; cycles \; of \; length \; l_2$$
$$s = \frac{i}{2} \;\rightarrow\; s_{\frac{i-1}{2}} \; cycles \; of \; length \; l_{\frac{i-1}{2}}$$

where $(s_t)(l_t) = k \; for \; 1 \le t \le \frac{i-1}{2}$.

Since such graph is an EFL k – star, hence the total number of edges is $2\left[k\left(\frac{i-1}{2}\right)\right] + k = ki$. Thus, the number of bridge points is $\frac{ki}{k} = i$.

Case III. When *i* is odd and *k* is even.
From a com-bridge point transformation of an EFL graph, construct such $(k \times i) - EFL$ by joining the components G_j and $G_{\left(j+\frac{k}{2}\right) mod\, k}$ and G_j and $G_{(j+s) mod\, k}$ where $1 < s \le \left\lfloor \frac{i}{2} \right\rfloor$.

$$s = 2 \;\rightarrow\; s_1 \; cycles \; of \; length \; l_1$$
$$s = 3 \;\rightarrow\; s_2 \; cycles \; of \; length \; l_2$$
$$\vdots$$
$$s = \frac{i}{2} \;\rightarrow\; s_{\frac{i}{2}-1} \; cycles \; of \; length \; l_{\frac{i}{2}-1}$$

where $(s_t)(l_t) = k \; for \; 1 \le t \le \frac{i}{2} - 1$.

Since such graph is an EFL k – star, hence the total number of edges is $2\left[k\left(\frac{i-1}{2}\right)\right] + k = ki$. Thus, the number of bridge points is $\frac{ki}{k} = i$.

Proposition 5:
The com-bridge transformation of a $(k \times i) - EFL$ graph, where $k - i = 1 \; for \; i \ge 3$, is a complete graph on k vertices.

Proof:
Let G be a $(k \times i) - EFL$ graph where $k - i = 1 \; for \; i \ge 3$, that is, G has k components with exactly i bridge points. By the hypothesis, $k - i = 1$ implies $i = k - 1$. Hence, G has k components with exactly $k - i$ vertices each. Therefore, G is a complete graph on k vertices.

Proposition 6:
If G is a $(k \times i) - EFL$ graph, then G is k colorable.

Proof:
Since every component of a $(k \times i) - EFL$ graph is a complete graph on k vertices, then each k components is k colorable, that is, every G_j is k-colourable. Now, since $i < k$, hence a greedy coloring can be performed to color the i bridge points of each component from the available k colors and the remaining vertices of each component which are not bridge point/s can be colored properly by the remaining $k - i$ colors. Thus, it is not possible to color G by $k + 1$ colors.

Definition:
A ***k* – *fan*** graph is an EFL graph with k components, $k - 1$ of which all share a common bridge point, and each exactly one other bridge point.

Some examples of ***k* – *fan*** graphs.

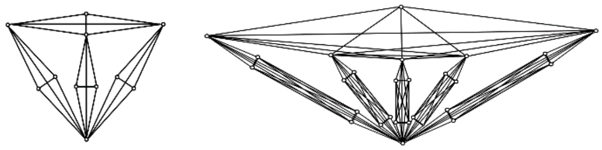

Figure 4: EFL k – fan graphs

It is clear from the definition above that since $k - 1$ components share a common bridge point, hence each of them should be connected to the k^{th} component for these $k - 1$ components have exactly two bridge points. Thus, the k^{th} component has $k - 1$ bridge points.

Proposition 7:
A ***k* – *fan*** graph G is k colourable.

Proof:
Let G be a *k* – *fan* graph. It implies that the k^{th} component has exactly $k - 1$ bridge points and the k^{th} component is k colourable. Hence color these $k - 1$ bridge points from the available k colors and color the common bridge point of the $k - 1$ components by the remaining 1 color of the k^{th} component. Thus, each of the two bridge points of the $k - 1$ components are colored already. So, color the remaining vertices of the $k - 1$ components (*which are not bridge points*) by the remaining $k - 2$ available colors of each component.

References

[1]. Bondy, J. A. and U. S. R. Murty, "Graph Theory". Springer 2008

[2]. Chang, W. I. and E. Lawler, "Edge Coloring on Hypergraphs and Conjecture of Erdös, Faber, Lovasz", Combinatorica 8, pp. 293 – 295, 1988

[3]. Erdös, Paul, "On the combinatorial problems which I most like to see solved.", Combinatorica 1, pp. 25 – 42, 1981

[4]. Horak, Peter, "A Coloring Problem Related to the Erdös-Faber-Lovasz Conjecture.", Journal of Combinatorial Theory, Series B 50, 323, pp. 321 – 322, 1990.

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