A Cauchy Problem For A Class Of Nonlinear Hyperbolic First Order Partial Differential Equation In A Banach Space

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Abstract: The ultimate work done in this research is the vast and active field of nonlinear hyperbolic first order partial differential equations. A class of nonlinear hyperbolic first order partial differential equation in a Banach space was investigated by converting it to Cauchy-like equation, than show that the operator A is accretive, M-accretive and thus admit a solution.

Keywords: Accretive, Cauchy Problem, Non-linear PDE, Hyperbolic

1.0 Introduction

The distinction between linear and nonlinear PDEs is extremely important in computational science. Many linear PDE problems can be solved exactly using techniques such as separation of variables, superposition, Fourier series, Laplace transform and Fourier transform. Exact solutions are valuable in a computational setting because they can be used to assist the computational scientist in the often difficult exercise of code validation. Generally, nonlinear PDEs do not yield to analytical solution approaches. Since most leading edge work in computational science involves nonlinear PDEs, a great deal of effort is directed toward obtaining numerical solutions. Whenever possible, computational scientists draw from the field of linear PDEs for guidance and insight in developing numerical methods for the more difficult nonlinear PDEs. In addition to the distinction between linear and nonlinear PDEs, it is important for the computational scientist to know that there are different classes of PDEs. Just as different solution techniques are called for in the linear versus the nonlinear case, different numerical methods are required for the different classes of PDEs, whether the PDE is linear or nonlinear. The need for this specialization in numerical or analytical approach is rooted in the physics from which the different classes of PDEs arise. By analogy with the conic sections (ellipse, parabola and hyperbola) partial differential equations have been classified as elliptic, parabolic and hyperbolic. Just as an ellipse is a smooth, rounded object, solutions to elliptic equations tend to be quite smooth. Elliptic equations generally arise from a physical problem that involves a diffusion process that has reached equilibrium, a steady state temperature distribution, for example. The hyperbola is the disconnected conic section. By analogy, hyperbolic equations are able to support solutions with discontinuities, for example a shock wave. Hyperbolic PDEs usually arise in connection with mechanical oscillators, such as a vibrating string, or in convection driven transport problems. Mathematically, parabolic PDEs serve as a transition from the hyperbolic PDEs to the elliptic PDEs. Physically, parabolic PDEs tend to arise in time dependent diffusion problems, such as the transient flow of heat in accordance with Fourier's law of heat conduction. The ultimate work done in this research is the vast and active field of nonlinear hyperbolic first order partial differential equations. Leaving aside quantum mechanics, which remains to date an inherently linear theory, most real-world physical systems, including gas dynamics, fluid mechanics, elasticity, relativity, ecology, neurology, thermodynamics, and many more, are modeled by nonlinear partial differential equations. Attempts to survey, in such a small space, even a tiny fraction of such an all-encompassing range of phenomena, methods, results, and mathematical developments, are doomed to failure.

> *"Some methods of constructing global weak solutions of the Cauchy problem are known for conservation laws with initial data of a fairly general type. For example, smoothing methods and finite-difference schemes for solving initial value problems.*
>
> *General non-linear hyperbolic equations and systems can be reduced to quasi-linear first-order hyperbolic systems by differentiating with respect to the independent variables. Any second-order hyperbolic equation can be reduced to a first-order symmetric hyperbolic system, and the facts relating to first-order hyperbolic systems remain valid also for a single second-order hyperbolic equation".*[1]

In the early twentieth century, an existence theorem for the solution of the Cauchy problem for a single hyperbolic equation of higher order was obtained under the requirement of a fairly high smoothness of the coefficients of the equation [2]. According [1] the Cauchy problem for hyperbolic quasi-linear equations of higher order has been studied by reducing it to a similar problem for quasi-linear partial differential equation of order one. [3], investigated a class of quasi-linear hyperbolic first order partial differential equation (PDE) in a Banach space, by converting it to Cauchy-like problem and thereafter established that it is m-accretive. [4] Used the same technique on a class of quasi-linear parabolic partial differential equations before arriving at the fact that it is also m-accretive. Using some results established in [3] and [5] investigated on a non-linear parabolic partial differential equation in a closed, bounded and continuous domain by converting such an equation into an abstract Cauchy problem, this operator was shown to be m-accretive thus establishing that this partial differential equation has a solution by the fundamental results of [6] on the theory of accretive operators. [7] assumed if X be a real Banach space, and X^* its conjugate space with the pairing between w in X^* and u in X of the form (w, u), then; a mapping T from X to 2^{X^*} is said to be accretive if for each u and v in X, x in $T(u)$, y in $T(v)$,

$$(x - y, J(u - v)) \geq 0,$$

where J is a duality mapping of X into X^*; i.e., J is a mapping of X into X^* such that for each u in X, $\|J(u)\| = \|u\|$ and $(Ju, u) = \|u\|^2$. (For single-valued mappings, this is equivalent to the form

$$(T(u) - T(v), J(u - v)) \geq 0,$$

for all u and v in $D(T)$, where $D(T)$ is the domain of T).

T is said to be maximal accretive if it is not properly contained in another accretive mapping from X to 2^{X^*}. T is said to be hypermaximal accretive if T is accretive and if $R(T + I)$, the range of $(T + I)$ is all of X (I = the identity mapping of X). It is equivalent to the statement that for all $\lambda > 0$, $(I + \lambda T)$ is a well-defined Lipchitzian mapping of X into X with Lipchitz constant ≤ 1.

A mapping T of X into 2^{X^*} is said to be coaccretive if for each u and v in X, x in $T(u)$, y in $T(v)$, we have

$$(u - v, J(u - v)) \geq 0,$$

(For single-valued mappings, this becomes $(u - v, J(T(u) - T(v))) \geq 0$, for all u and v in $D(T)$.)

Obviously T is co-accretive if and only if T^{-1} is accretive.

If T is a mapping from X to X, T is said to be $\phi - accretive$ if there exists a covering of X by neighborhoods N such that for each neighborhood, there exists a mapping ϕ_N (not necessarily continuous) of $N \times N$ into X^* satisfying

(i) For all u and v in N,

$(T(u) - T(v), \phi(u, v)) \geq 0$,

(ii) There exists a function $C : R^+ \to R^+$ that is continuous, positive, and non-increasing for $r \geq 0$ with $\int c(r)dr = \infty$ (with c independent of the choice of N), such that for all u and v in N, and r_o = max($\|u\|, \|v\|$),

$$\|\phi_N(u, v)\| \leq \|u - v\|;$$

$$(u - v, \phi(u, v)) \geq c(\|r_0\|)\|u - v\|^2.$$

If we consider a differential equation $\frac{du}{dt} + Au(t) = 0$, that describes an evolution system, such $A : X \to X$ is nonlinear accretive map, then at equilibrium state, $\frac{du}{dt} = 0$ and so a solution to $Au(t) = 0$ describes the equilibrium or stable state of the system. This is very desirable in many applications, such as physics, economics, ecology etc. Therefore considering the role that $Au(t) = 0$ plays in the system, and that generally A is nonlinear, research efforts have been developed to determine methods of solving the equation. But unfortunately due to the nonlinearity property of A, rarely closed from solutions of the equation occur.

This study is aiming at extending [3] work on a Cauchy problem for a quasi-linear hyperbolic first order partial differential equation (PDE), to a Cauchy problem for a class of nonlinear hyperbolic first order partial differential equation (PDE) in a Banach space, using some idea established by [5].

2.0 Methodology

Let X be a real Banach space with norm $\| \cdot \|$ and dual X^*.

An operator A with domain $D(A)$ and range $R(A)$, in X is said to be accretive, if for all $x_1, x_2 \in D(A)$ and $r > 0$, there holds the inequality [6].

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + (Ax_1 - Ax_2)\| \qquad (1)$$

An accretive operator A is said to be m-accretive if $R(I + rA) = X$ for all $r > 0$, where I is the identity operator on X. In terms of the concept of contractions, an operator A is said to be accretive if $R(I + rA)^{-1}$ is a contraction for $r \geq 0$ that is if

$$\|x_1 + rAx_1 - (x_2 + rAx_2)\| \geq \|x_1 - x_2\|. \qquad (2)$$

If X is a Hilbert space, the accretive condition (2) reduces to

$\operatorname{Re}\langle Ax_1 + Ax_2, x_1 + x_2\rangle \geq 0$ for all $x_1, x_2 \in X$ (3)

2.1 Lemma [4]
Let X be a real Banach space and let $A : D(A) \subset X \to X$ be an m-accretive operator such $u = (A + n^{-1}I)x_1, v = (A + n^{-1}I)x_2$

Where $u, v \in X; x_1, x_2 \in D(A); n \in R^+$

Then operator $(A + n^{-1}I) : X \to D(A)$ is continuous and bounded.

3.0 RESULTS

Existence of Solution of Non-Linear Hyperbolic Equation
Considering a nonlinear hyperbolic first order partial differential equation defined on a closed, bounded and continuous domain (Space of $C[0,1]$).

$$\phi(u_t) - u_x = 0, \qquad 0 < x < 1 \qquad (4)$$

$$u(0, x) = u_0(x), \qquad 0 < x < 1$$

$$u(t,0) = u(t,1) = 0 \qquad t > 0,$$

Where $\phi : R \to R$, is continuous, strictly increasing and $\phi(0) = 0$, which we transform into the initial value problem

$$\frac{du}{dt} + Au = 0, \quad u(0) = u_0 \qquad (5)$$

defined on a Banach space $X = C[0,1]$ assuming that u_x is continuous, strictly increasing, $u_{xx}(0) = 0$, $u_x(R) = R$ and consequently A is m-accretive.

3.1 Theorem:
Let $X = L^1([0,1])$, $D(A) = \{u \in C[0,1] : u, u'$ are continuous and $u(0) = u(1) = 0$ $\}$ and $A(u) = \{v \in C[0,1] : \phi(-v) = u'\}$, for $u \in D(A)$; where $v = u_t$, then A is m-accretive.

Remark; The operator A acts in accordance with Partial Differential Equation and $D(A)$ consists of functions satisfying boundary condition

Proof:
First, it is good to show that the operator A is accretive. Since $\phi : R \to R$, is continuous, strictly increasing with $\phi(0) = 0$, we define the associated operator A as a single-valued function.

Suppose $u_1, u_2 \in D(A)$, and $v_1 \in Au_1, v_2 \in Au_2$

Let $u_1 \neq u_2$ then

$$\|u_1 - u_2\|_\infty = \max_{0 \le x \le 1} |u_1(x) - u_2(x)|$$
$$= u_1(x_0) - u_2(x_0)$$

Since $u_1 = u_2 = 0$ at $x = 0$ and $x = 1 \Rightarrow 0 < x_0 < 1$

Case 1:

If $v_2(x_0) \le v_1(x_0)$ and $\lambda > 0$ then

$$\max_{0 \le x \le 1} |u_1(x) - u_2(x) + \lambda(v_1(x) - v_2(x)|$$
$$\ge (u_1(x_0) - u_2(x_0) + \lambda(v_1(x_0) - v_2(x_0))$$

Case 2:

If $v_2(x_0) > v_1(x_0)$

$$-v_2(x_0) < -v_1(x_0)$$

$$\Leftrightarrow -v_1(x_0) > -v_2(x_0)$$

Therefore by the monotonicity of ϕ

$\Rightarrow \phi(-v_2(x)) \le \phi(-v_1(x))$ on $[x_0 - \delta, x_0 + \delta]$ where $\delta > 0$

Consequently $(u_1 - u_2)'(x) = \phi(-v_1(x) - \phi(-v_2(x)) \ge 0$ on $[x_0 - \delta, x_0 + \delta]$

Hence by [5], $u_1 - u_2$ is convex on $[x_0 - \delta, x_0 + \delta]$ and therefore assumes its maximum at the interior point x_0. Then $u_1 - u_2$ is constant $x_0 - \delta \le x \le x_0 + \delta$. If x_0 is the least number in $[0,1]$ for which case (1) holds leads to a contradiction that $v_2(x_0) > v_1(x_0) \Leftrightarrow u_1 - u_2 < 0$.

Hence $v_1(x_0) \ge v_2(x_0)$.

Using definition of accretive and the concepts of accretive operator in condition (1) then case 1 and 2 shows that A is accretive. Now it remains to show that it is m-accretive, that is, $R(I + A) = C([0,1])$, where I is the identity operator, R is range, and r is assumed to equal to 1. Consider, $h \in C[0,1]$, $u \in D(A)$ such that $h \in (I + A)u$ or $h - u = Au$. for $u = \phi(v)$ and $\beta(u) = \phi^{-1}$ means that the setting $u(0) = u(1) = 0$ and $u' = \phi(u - h)$ can be solved. Since A is accretive then if $\beta(u) \in D(A)$ and $\beta(u) = u' - h$, then,

$$\|\beta(u)\|_\infty \le \|\beta(u) - u' - \beta(u) - 0\|_\infty = \|h\|_\infty = \max_{0 \le u \le 1} |h|$$

Also by the mean value theorem, for some $\xi \in [0,1], u'(\xi) = 0$ since $u(0) = u(1) = 0$

$$|u'(x)| \le \left|\int_\xi^x u'(s)ds\right| x$$

$$\le \int_0^1 |\beta(u) - h| dx \le 2\|h\|_{C[0,1]} \text{ and}$$

$$|u(x)| \le \left|\int_0^x u'(s)ds\right|$$

$$\le \int_0^1 |u'(s)| ds \le 2\|h\|_{C[0,1]} = \max_{0 \le x \le 1} |h|$$

Since the operator A has been shown to be accretive, and since $R(I + A) = C[0,1]$, then it is m-accretive. Hence by [6], this operator A for a first order nonlinear hyperbolic problem admits a solution. Thus, a solution for this non-linear hyperbolic first order partial differential equation problem.

This ends the proof.

4.0 Discussion of the Results

Here, $X = L^1[0,1]$ is the space of closed bounded and continuous domain, we define the domain of the non-linear operator as a dense subset of X; under the conditions given in Preposition, our non-linear operator is not only accretive but also m-accretive. It has been established that the non-linear partial differential equation of the type in equation (4), has a Cauchy-like equation (5) where the operator which is defined in the space of $X = L^1[0,1]$ is m-accretive. It is known that if X is a Banach space (Hilbert space), then an accretive mapping of X into itself is locally bounded at every point of the interior of its domain. We had shown that this is also true in case 1 and 2 i.e the proof of **preposition** provided that the duality mapping of X is weakly continuous. We used this fact to show that, under some stated conditions, an accretive mapping is m-accretive if and only if it is maximally accretive i.e. the $R(I + rA) = C[0,1]$, where $r > 0$.

5.0 Conclusion

The m-accretiveness of the non-linear operator in space of Lipchitz continuous domain problem has been established, just as it was observed in the hyperbolic case as by [3]. We have established in this work that, a nonlinear hyperbolic partial differential equation of the kind in equation (4) defined on a Banach space $X = L^1[0,1]$ is continuous, accretive and thus admit a solution. The analysis of this non-linear hyperbolic partial differential equation problem for accretive properties will no doubt help mathematics educators, researchers and scientists who are interested in nonlinear systems and nonlinear partial differential equations.

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